Lecture Notes in Lie Groups

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Abstract

These lecture notes in Lie Groups are designed for a 1–semester third year or graduate course in mathematics, physics, engineering, chemistry or biology. This landmark theory of the 20th Century mathematics and physics gives a rigorous foundation to modern dynamics, as well as field and gauge theories in physics, engineering and biomechanics. We give both physical and medical examples of Lie groups. The only necessary background for comprehensive reading of these notes are advanced calculus and linear algebra.

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1 Preliminaries: Sets, Maps and Diagrams

1.1 Sets

Given a map (or, a function) $f : A \to B$, the set A is called the *domain* of f, and denoted $\mathrm{Dom}\, f$. The set B is called the *codomain* of f, and denoted $\mathrm{Cod}\, f$. The codomain is not to be confused with the *range* of $f(A)$, which is in general only a subset of B (see [8, 9]).

A map $f : X \to Y$ is called *injective*, or 1–1, or an *injection*, iff for every y in the codomain Y there is *at most* one x in the domain X with $f(x) = y$. Put another way, given x and x' in X, if $f(x) = f(x')$, then it follows that $x = x'$. A map $f : X \to Y$ is called *surjective*, or *onto*, or a *surjection*, iff for every y in the codomain $\mathrm{Cod}\, f$ there is *at least* one x in the *domain* X with $f(x) = y$. Put another way, the *range* $f(X)$ is equal to the codomain Y. A map is *bijective* iff it is both injective and surjective. Injective functions are called *monomorphisms*, and surjective functions are called *epimorphisms* in the *category of sets* (see below). Bijective functions are called *isomorphisms*.

A *relation* is any subset of a *Cartesian product* (see below). By definition, an *equivalence relation* α on a set X is a relation which is *reflexive, symmetrical* and *transitive*, i.e., relation that satisfies the following three conditions:

1. *Reflexivity*: each element $x \in X$ is equivalent to itself, i.e., $x\alpha x$;

2. *Symmetry*: for any two elements $a, b \in X$, $a\alpha b$ implies $b\alpha a$; and

3. *Transitivity*: $a\alpha b$ and $b\alpha c$ implies $a\alpha c$.

Similarly, a relation $\leq$ defines a *partial order* on a set S if it has the following properties:

1. *Reflexivity*: $a \leq a$ for all $a \in S$;

2. *Antisymmetry*: $a \leq b$ and $b \leq a$ implies $a = b$; and

3. *Transitivity*: $a \leq b$ and $b \leq c$ implies $a \leq c$.

A *partially ordered set* (or *poset*) is a set taken together with a partial order on it. Formally, a partially ordered set is defined as an ordered pair $P = (X, \leq)$, where X is called the *ground set* of P and $\leq$ is the partial order of P.

1.2 Maps

Let f and g be maps with domains A and B. Then the maps $f+g$, $f-g$, fg, and f/g are defined as follows (see [8, 9]):

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) && \text{domain } = A\cap B,\\
(f-g)(x) &= f(x)-g(x) && \text{domain } = A\cap B,\\
(fg)(x) &= f(x)\,g(x) && \text{domain } = A\cap B,\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{domain } = \{x\in A\cap B : g(x)\neq 0\}.
\end{aligned}$$

Given two maps f and g, the composite map $f\circ g$, called the *composition* of f and g, is defined by

$$(f\circ g)(x) = f(g(x)).$$

The $(f\circ g)-$machine is composed of the $g-$machine (first) and then the $f-$machine,

$$x\to [[g]]\to g(x)\to [[f]]\to f(g(x)).$$

For example, suppose that $y=f(u)=\sqrt{u}$ and $u=g(x)=x^2+1$. Since y is a function of u and u is a function of x, it follows that y is ultimately a function of x. We calculate this by substitution

$$y=f(u)=f\circ g=f(g(x))=f(x^2+1)=\sqrt{x^2+1}.$$

If f and g are both differentiable (or smooth, i.e., C^∞) maps and $h=f\circ g$ is the composite map defined by $h(x)=f(g(x))$, then h is differentiable and h' is given by the product:

$$h'(x)=f'(g(x))\,g'(x).$$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$ are both differentiable maps, then

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}.$$

The reason for the name *chain rule* becomes clear if we add another link to the chain. Suppose that we have one more differentiable map $x=h(t)$. Then, to calculate the derivative of y with respect to t, we use the chain rule twice,

$$\frac{dy}{dt}=\frac{dy}{du}\frac{du}{dx}\frac{dx}{dt}.$$

Given a 1–1 continuous (i.e., C^0) map F with a nonzero *Jacobian* $\left|\frac{\partial(x,\ldots)}{\partial(u,\ldots)}\right|$ that maps a region S onto a region R, we have the following substitution formulas:

1. For a single integral,

$$\int_R f(x)\,dx = \int_S f(x(u))\frac{\partial x}{\partial u}du;$$

2. For a double integral,

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv;$$

3. For a triple integral,

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw;$$

4. Generalization to $n-$tuple integrals is obvious.

1.3 Commutative Diagrams

Many properties of mathematical systems can be unified and simplified by a presentation with *commutative diagrams of arrows*. Each arrow $f : X \to Y$ represents a function (i.e., a map, transformation, operator); that is, a source (domain) set X, a target (codomain) set Y, and a rule $x \mapsto f(x)$ which assigns to each element $x \in X$ an element $f(x) \in Y$. A typical diagram of sets and functions is (see [8, 9]):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & f(X) \arrow[d, "g"] \\ & g(f(X)) \end{tikzcd}$$

This diagram is *commutative* iff $h = g \circ f$, where $g \circ f$ is the usual composite function $g \circ f : X \to Z$, defined by $x \mapsto g(f(x))$.

Less formally, composing maps is like following directed paths from one object to another (e.g., from set to set). In general, a diagram is commutative iff any two paths along arrows that start at the same point and finish at the same point yield the same 'homomorphism' via compositions along successive arrows. Commutativity of the whole diagram follows from commutativity of its triangular components. Study of commutative diagrams is popularly called 'diagram chasing', and provides a powerful tool for mathematical thought.

Many properties of mathematical constructions may be represented by *universal properties* of diagrams. Consider the *Cartesian product* $X \times Y$ of two sets, consisting as usual of all ordered pairs $\langle x, y\rangle$ of elements $x \in X$ and $y \in Y$. The projections

$\langle x, y\rangle \mapsto x$, $\langle x, y\rangle \mapsto y$ of the product on its 'axes' X and Y are functions $p : X \times Y \to X$, $q : X \times Y \to Y$. Any function $h : W \to X \times Y$ from a third set W is uniquely determined by its composites $p \circ h$ and $q \circ h$. Conversely, given W and two functions f and g as in the diagram below, there is a unique function h which makes the following diagram commute:

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "p"] \arrow[r, "q"'] & Y \end{tikzcd}$$

This property describes the Cartesian product $X \times Y$ uniquely; the same diagram, read in the category of topological spaces or of groups, describes uniquely the Cartesian product of spaces or of the direct product of groups.

2 Groups

A *group* is a pointed set (G, e) with a *multiplication* $\mu : G \times G \to G$ and an *inverse* $\nu : G \to G$ such that the following diagrams commute (see [10, 8, 9]):

1. $$\begin{tikzcd} G \arrow[r, "{(e,1)}"] \arrow[dr, "1"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "{(1,e)}"'] \arrow[dl, "1"] \\ & G & \end{tikzcd}$$

   (e is a two–sided identity)

2. $$\begin{tikzcd} G \times G \times G \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

   (associativity)

3.

$$
\begin{tikzcd}
G \arrow[r, "{(\nu,1)}"] \arrow[dr, "e"'] & G \times G \arrow[r, "{(1,\nu)}"] \arrow[d, "\mu"] & G \\
 & G \arrow[ur, "e"'] &
\end{tikzcd}
$$

(inverse).

Here $e : G \to G$ is the constant map $e(g) = e$ for all $g \in G$. $(e, 1)$ means the map such that $(e, 1)(g) = (e, g)$, etc. A group G is called *commutative* or *Abelian group* if in addition the following diagram commutes

$$
\begin{tikzcd}
G \times G \arrow[rr, "T"] \arrow[dr, "\mu"'] & & G \times G \arrow[dl, "\mu"] \\
 & G &
\end{tikzcd}
$$

where $T : G \times G \to G \times G$ is the switch map $T(g_1, g_2) = (g_2, g_1)$, for all $(g_1, g_2) \in G \times G$.

A group G *acts* (on the left) on a set A if there is a function $\alpha : G \times A \to A$ such that the following diagrams commute:

1.

$$
\begin{tikzcd}
A \arrow[r, "{(e,1)}"] \arrow[dr, "1"'] & G \times A \arrow[d, "\alpha"] \\
 & A
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}
G \times G \times A \arrow[r, "1 \times \alpha"] \arrow[d, "\mu \times 1"'] & G \times A \arrow[d, "\alpha"] \\
G \times A \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

where $(e, 1)(x) = (e, x)$ for all $x \in A$. The *orbits* of the action are the sets $Gx = \{gx : g \in G\}$ for all $x \in A$.

Given two groups $(G, *)$ and $(H, \cdot)$, a *group homomorphism* from $(G, *)$ to $(H, \cdot)$ is a function $h : G \to H$ such that for all x and y in G it holds that

$$h(x * y) = h(x) \cdot h(y).$$

From this property, one can deduce that h maps the identity element e_G of G to the identity element e_H of H, and it also maps inverses to inverses in the sense that $h(x^{-1}) = h(x)^{-1}$. Hence one can say that h is *compatible* with the *group structure*.

The *kernel* $\operatorname{Ker} h$ of a group homomorphism $h : G \to H$ consists of all those elements of G which are sent by h to the identity element e_H of H, i.e.,

$$\operatorname{Ker} h = \{x \in G : h(x) = e_H\}.$$

The *image* $\operatorname{Im} h$ of a group homomorphism $h : G \to H$ consists of all elements of G which are sent by h to H, i.e.,

$$\operatorname{Im} h = \{h(x) : x \in G\}.$$

The kernel is a *normal subgroup* of G and the image is a *subgroup* of H. The homomorphism h is *injective* (and called a *group monomorphism*) iff $\operatorname{Ker} h = e_G$, i.e., iff the kernel of h consists of the identity element of G only.

3 Manifolds

A *manifold* is an abstract mathematical space, which locally (i.e., in a close–up view) resembles the spaces described by *Euclidean geometry*, but which globally (i.e., when viewed as a whole) may have a more complicated structure (see [11]). For example, the *surface of Earth* is a manifold; locally it seems to be flat, but viewed as a whole from the outer space (globally) it is actually round. A manifold can be constructed by 'gluing' separate *Euclidean spaces* together; for example, a world map can be made by gluing many maps of local regions together, and accounting for the resulting distortions.

As main pure–mathematical references for manifolds we recommend popular graduate textbooks by two ex–*Bourbaki* members, *Serge Lang* [13, 12] and Jean Dieudonne [14, 15]. Besides, the reader might wish to consult some other 'classics', including [11, 16, 17, 18, 19, 20, 3]. Finally, as first–order applications, we recommend three popular textbooks in mechanics, [2, 1, 4], as well as our own geometrical monographs [8, 9].

Another example of a manifold is a *circle* S^1. A small piece of a circle appears to be like a slightly–bent part of a straight line segment, but overall the circle and the

segment are different 1D manifolds. A circle can be formed by bending a straight line segment and gluing the ends together.[1]

The surfaces of a *sphere*[2] and a *torus*[3] are examples of 2D manifolds. Manifolds

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[1]Locally, the circle looks like a line. It is 1D, that is, only one coordinate is needed to say where a point is on the circle locally. Consider, for instance, the top part of the circle, where the $y-$coordinate is positive. Any point in this part can be described by the $x-$coordinate. So, there is a continuous *bijection* χ_{top} (a mapping which is 1–1 both ways), which maps the top part of the circle to the open interval $(-1, 1)$, by simply projecting onto the first coordinate: $\chi_{top}(x, y) = x$. Such a function is called a *chart*. Similarly, there are charts for the bottom, left , and right parts of the circle. Together, these parts *cover* the whole circle and the four charts form an *atlas* (see the next subsection) for the circle. The top and right charts overlap: their intersection lies in the quarter of the circle where both the $x-$ and the $y-$coordinates are positive. The two charts χ_{top} and χ_{right} map this part bijectively to the interval $(0, 1)$. Thus a function T from $(0, 1)$ to itself can be constructed, which first inverts the top chart to reach the circle and then follows the right chart back to the interval:

$$T(a) = \chi_{\mathrm{right}}\left(\chi_{\mathrm{top}}^{-1}(a)\right) = \chi_{\mathrm{right}}\left(a, \sqrt{1-a^2}\right) = \sqrt{1-a^2}.$$

Such a function is called a *transition map*. The top, bottom, left, and right charts show that the circle is a manifold, but they do not form the only possible atlas. Charts need not be geometric projections, and the number of charts is a matter of choice. T and the other transition functions are differentiable on the interval $(0, 1)$. Therefore, with this atlas the circle is a *differentiable,* or *smooth manifold.*

[2]The surface of the sphere S^2 can be treated in almost the same way as the circle S^1. It can be viewed as a subset of $\mathbb{R}^3$, defined by: $S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. The sphere is 2D, so each chart will map part of the sphere to an open subset of $\mathbb{R}^2$. Consider the northern hemisphere, which is the part with positive z coordinate. The function χ defined by $\chi(x, y, z) = (x, y)$, maps the northern hemisphere to the open unit disc by projecting it on the $(x, y)-$plane. A similar chart exists for the southern hemisphere. Together with two charts projecting on the $(x, z)-$plane and two charts projecting on the $(y, z)-$plane, an atlas of six charts is obtained which covers the entire sphere. This can be easily generalized to an nD sphere $S^n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_1^2 + x_2^2 + ... + x_n^2 = 1\}$.

An $n-$sphere S^n can be also constructed by gluing together two copies of $\mathbb{R}^n$. The transition map between them is defined as $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} : x \mapsto x/\|x\|^2$. This function is its own inverse, so it can be used in both directions. As the transition map is a $(C^\infty)-$*smooth function*, this atlas defines a *smooth manifold*.

[3]A torus (pl. tori), denoted by T^2, is a doughnut–shaped surface of revolution generated by revolving a circle about an axis coplanar with the circle. The sphere S^2 is a special case of the torus obtained when the axis of rotation is a diameter of the circle. If the axis of rotation does not intersect the circle, the torus has a hole in the middle and resembles a ring doughnut, a hula hoop or an inflated tire. The other case, when the axis of rotation is a chord of the circle, produces a sort of squashed sphere resembling a round cushion.

A torus can be defined parametrically by:

$$x(u, v) = (R + r\cos v)\cos u, \qquad y(u, v) = (R + r\cos v)\sin u, \qquad z(u, v) = r\sin v,$$

where $u, v \in [0, 2\pi]$, R is the distance from the center of the tube to the center of the torus, and r is the radius of the tube. According to a broader definition, the generator of a torus need not be a circle but could also be an ellipse or any other conic section.

Topologically, a torus is a closed surface defined as product of two circles: $T^2 = S^1 \times S^1$. The

are important objects in mathematics, physics and control theory, because they allow more complicated structures to be expressed and understood in terms of the well–understood properties of simpler Euclidean spaces (see [9]).

The *Cartesian product* of manifolds is also a manifold (note that not every manifold can be written as a product). The dimension of the product manifold is the sum of the dimensions of its factors. Its topology is the product topology, and a Cartesian product of charts is a chart for the product manifold. Thus, an atlas for the product manifold can be constructed using atlases for its factors. If these atlases define a differential structure on the factors, the corresponding atlas defines a differential structure on the product manifold. The same is true for any other structure defined on the factors. If one of the factors has a boundary, the product manifold also has a boundary. Cartesian products may be used to construct tori and cylinders, for example, as $S^1 \times S^1$ and $S^1 \times [0,1]$, respectively.

Manifolds need not be *connected* (all in 'one piece'): a pair of separate circles is also a *topological manifold*(see below). Manifolds need not be *closed*: a line segment without its ends is a manifold. Manifolds need not be *finite*: a parabola is a topological manifold.

Manifolds[4] can be viewed using either extrinsic or intrinsic view. In the *extrinsic view*, usually used in geometry and topology of surfaces, an nD manifold M is seen as embedded in an $(n+1)$D Euclidean space $\mathbb{R}^{n+1}$. Such a manifold is called a 'codimension 1 space'. With this view it is easy to use intuition from Euclidean spaces to define additional structure. For example, in a Euclidean space it is always clear whether a vector at some point is tangential or normal to some surface through that point. On the other hand, the *intrinsic view* of an nD manifold M is an abstract way of considering M as a topological space by itself, without any need

surface described above, given the relative topology from $\mathbb{R}^3$, is *homeomorphic* to a topological torus as long as it does not intersect its own axis.

One can easily generalize the torus to arbitrary dimensions. An $n-$torus T^n is defined as a product of n circles: $T^n = S^1 \times S^1 \times \cdots \times S^1$. Equivalently, the $n-$torus is obtained from the $n-$cube (the $\mathbb{R}^n-$generalization of the ordinary cube in $\mathbb{R}^3$) by gluing the opposite faces together.

An $n-$torus T^n is an example of an nD *compact manifold*. It is also an important example of a *Lie group* (see below).

[4]

Additional structures are often defined on manifolds. Examples of manifolds with additional structure include:

- *differentiable* (or, *smooth manifolds*, on which one can do calculus;
- *Riemannian manifolds*, on which *distances* and *angles* can be defined; they serve as the *configuration spaces* in mechanics;
- *symplectic manifolds*, which serve as the *phase spaces* in mechanics and physics;
- 4D pseudo–Riemannian manifolds which model *space–time* in general relativity.

for surrounding $(n+1)$D Euclidean space. This view is more flexible and thus it is usually used in high–dimensional mechanics and physics (where manifolds used represent configuration and phase spaces of dynamical systems), can make it harder to imagine what a tangent vector might be.

3.1 Definition of a Manifold

Consider a set M (see Figure 1) which is a *candidate* for a manifold. Any point $x \in M$ has its *Euclidean chart*, given by a 1–1 and *onto* map $\varphi_i : M \to \mathbb{R}^n$, with its *Euclidean image* $V_i = \varphi_i(U_i)$. More precisely, a chart φ_i is defined by (see [8, 9])

$$\varphi_i : M \supset U_i \ni x \mapsto \varphi_i(x) \in V_i \subset \mathbb{R}^n,$$

where $U_i \subset M$ and $V_i \subset \mathbb{R}^n$ are open sets.

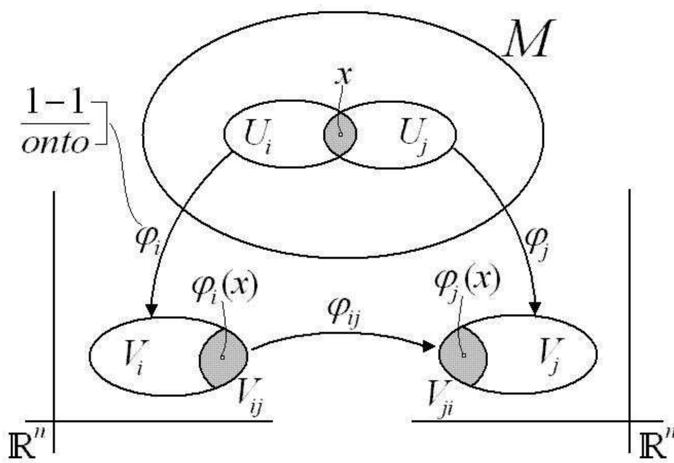

Figure 1: Geometric picture of the manifold concept.

Clearly, any point $x \in M$ can have several different charts (see Figure 1). Consider a case of two charts, $\varphi_i, \varphi_j : M \to \mathbb{R}^n$, having in their images two open sets, $V_{ij} = \varphi_i(U_i \cap U_j)$ and $V_{ji} = \varphi_j(U_i \cap U_j)$. Then we have *transition functions* φ_{ij} between them,

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : V_{ij} \to V_{ji}, \qquad \text{locally given by} \qquad \varphi_{ij}(x) = \varphi_j(\varphi_i^{-1}(x)).$$

If transition functions φ_{ij} exist, then we say that two charts, φ_i and φ_j are *compatible.* Transition functions represent a general (nonlinear) *transformations of coordinates*, which are the core of classical *tensor calculus.*

A set of compatible charts $\varphi_i : M \to \mathbb{R}^n$, such that each point $x \in M$ has its Euclidean image in at least one chart, is called an *atlas*. Two atlases are *equivalent* iff all their charts are compatible (i.e., transition functions exist between them), so their union is also an atlas. A *manifold structure* is a class of equivalent atlases.

Finally, as charts $\varphi_i : M \to \mathbb{R}^n$ were supposed to be 1-1 and onto maps, they can be either *homeomorphisms*, in which case we have a *topological* (C^0) manifold, or *diffeomorphisms*, in which case we have a *smooth* (C^k) manifold.

3.2 Formal Definition of a Smooth Manifold

Given a chart (U, φ), we call the set U a *coordinate domain*, or a coordinate neighborhood of each of its points. If in addition $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a *coordinate ball*. The map φ is called a (*local*) *coordinate map*, and the component functions $(x^1, ..., x^n)$ of φ, defined by $\varphi(m) = (x^1(m), ..., x^n(m))$, are called *local coordinates* on U [8, 9].

Two charts (U_1, φ_1) and (U_2, φ_2) such that $U_1 \cap U_2 \neq \varnothing$ are called *compatible* if $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_2 \cap U_1)$ are open subsets of $\mathbb{R}^n$. A family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of compatible charts on M such that the U_α form a *covering* of M is called an *atlas*. The maps $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$ are called the *transition maps*, for the atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$, so that we have a commutative triangle:

$$
\begin{tikzcd}
 & U_{\alpha\beta} \subseteq M \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\
\varphi_\alpha(U_{\alpha\beta}) \arrow[rr, "\varphi_{\alpha\beta}"'] & & \varphi_\beta(U_{\alpha\beta})
\end{tikzcd}
$$

An atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ for a manifold M is said to be a C^k*−atlas*, if all transition maps $\varphi_{\alpha\beta} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$ are of class C^k. Two C^k atlases are called C^k*−equivalent*, if their union is again a C^k−atlas for M. An equivalence class of C^k−atlases is called a C^k*−structure* on M. In other words, a smooth structure on M is a *maximal* smooth atlas on M, i.e., such an atlas that is not contained in any strictly larger smooth atlas. By a C^k*−manifold* M, we mean a topological manifold together with a C^k−structure and a chart on M will be a chart belonging to some atlas of the C^k−structure. Smooth manifold means C^∞−manifold, and the word '*smooth*' is used synonymously with C^∞.

Sometimes the terms 'local coordinate system' or 'parametrization' are used instead of charts. That M is not defined with any particular atlas, but with an equivalence class of atlases, is a mathematical formulation of the *general covariance* principle. Every suitable coordinate system is equally good. A Euclidean chart

may well suffice for an open subset of $\mathbb{R}^n$, but this coordinate system is not to be preferred to the others, which may require many charts (as with polar coordinates), but are more convenient in other respects.

For example, the atlas of an $n-$sphere S^n has two charts. If $N = (1, 0, ..., 0)$ and $S = (-1, ..., 0, 0)$ are the north and south poles of S^n respectively, then the two charts are given by the stereographic projections from N and S:

$$\begin{aligned}\varphi_1 &: S^n\backslash\{N\} \to \mathbb{R}^n, \varphi_1(x^1, ..., x^{n+1}) = (x^2/(1-x^1), \dots, x^{n+1}/(1-x^1)), \text{ and} \\ \varphi_2 &: S^n\backslash\{S\} \to \mathbb{R}^n, \varphi_2(x^1, ..., x^{n+1}) = (x^2/(1+x^1), \dots, x^{n+1}/(1+x^1)),\end{aligned}$$

while the overlap map $\varphi_2 \circ \varphi_1^{-1} : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$ is given by the diffeomorphism $(\varphi_2 \circ \varphi_1^{-1})(z) = z/||z||^2$, for z in $\mathbb{R}^n\backslash\{0\}$, from $\mathbb{R}^n\backslash\{0\}$ to itself.

Various *additional structures* can be imposed on $\mathbb{R}^n$, and the corresponding manifold M will inherit them through its covering by charts. For example, if a covering by charts takes their values in a *Banach space* E, then E is called the *model space* and M is referred to as a C^k-*Banach manifold* modelled on E. Similarly, if a covering by charts takes their values in a *Hilbert space* $\mathcal{H}$, then $\mathcal{H}$ is called the *model space* and M is referred to as a C^k-*Hilbert manifold* modelled on $\mathcal{H}$. If not otherwise specified, we will consider M to be an Euclidean manifold, with its covering by charts taking their values in $\mathbb{R}^n$.

For a Hausdorff C^k-manifold the following properties are equivalent: (i) it is paracompact; (ii) it is metrizable; (iii) it admits a Riemannian metric;[5] (iv) each connected component is separable.

3.3 Smooth Maps Between Smooth Manifolds

A map $\varphi : M \to N$ between two manifolds M and N, with $M \ni m \mapsto \varphi(m) \in N$, is called a *smooth map*, or C^k-map, if we have the following charting [8, 9]:

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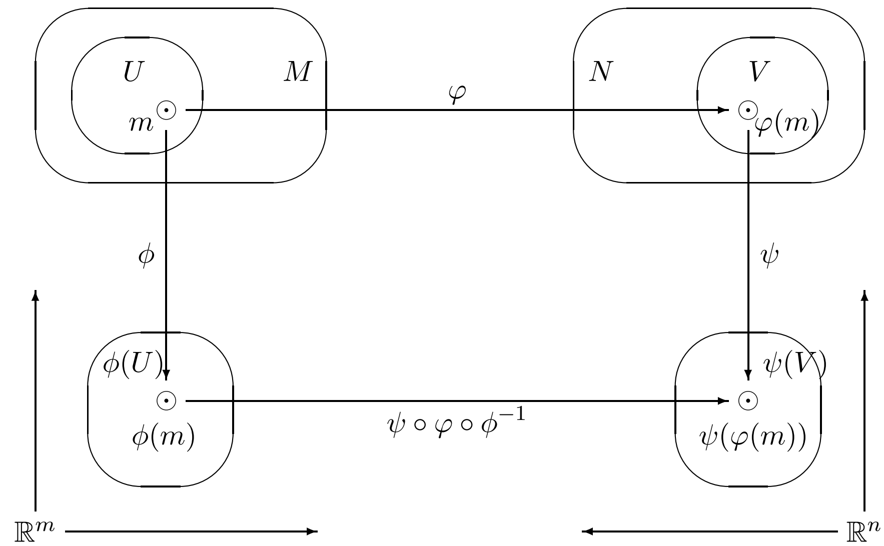
[5] Recall the corresponding properties of a *Euclidean metric* d. For any three points $x, y, z \in \mathbb{R}^n$, the following axioms are valid:

$$\begin{aligned}M_1 &: d(x,y) > 0, \text{ for } x \neq y; \qquad \text{and} \qquad d(x,y) = 0, \text{ for } x = y; \\ M_2 &: d(x,y) = d(y,x); \qquad\qquad M_3 : d(x,y) \leq d(x,z) + d(z,y).\end{aligned}$$



This means that for each $m \in M$ and each chart (V, ψ) on N with $\varphi(m) \in V$ there is a chart (U, ϕ) on M with $m \in U, \varphi(U) \subseteq V$, and $\Phi = \psi \circ \varphi \circ \phi^{-1}$ is C^k, that is, the following diagram commutes:

$$\begin{tikzcd}
M \supseteq U \arrow[r, "\varphi"] \arrow[d, "\phi"'] & V \subseteq N \arrow[d, "\psi"] \\
\phi(U) \arrow[r, "\Phi"'] & \psi(V)
\end{tikzcd}$$

Let M and N be smooth manifolds and let $\varphi : M \to N$ be a smooth map. The map φ is called a *covering*, or equivalently, M is said to *cover* N, if φ is surjective and each point $n \in N$ admits an open neighborhood V such that $\varphi^{-1}(V)$ is a union of disjoint open sets, each diffeomorphic via φ to V.

A C^k-map $\varphi : M \to N$ is called a C^k-*diffeomorphism* if φ is a bijection, $\varphi^{-1} : N \to M$ exists and is also C^k. Two manifolds are called diffeomorphic if there exists a diffeomorphism between them. All smooth manifolds and smooth maps between them form the category $\mathcal{M}$.

3.4 Tangent Bundle and Lagrangian Dynamics

The tangent bundle of a smooth $n-$manifold is the place where tangent vectors live, and is itself a smooth $2n-$manifold. Vector–fields are cross-sections of the tangent bundle. The *Lagrangian* is a natural energy function on the tangent bundle (see [8, 9]).

In mechanics, to each nD *configuration manifold* M there is associated its $2n$D *velocity phase–space manifold*, denoted by TM and called the tangent bundle of M (see Figure 2). The original smooth manifold M is called the *base* of TM. There is an onto map $\pi : TM \to M$, called the *projection*. Above each point $x \in M$ there is a *tangent space* $T_xM = \pi^{-1}(x)$ to M at x, which is called a *fibre*. The fibre $T_xM \subset TM$ is the subset of TM, such that the total tangent bundle, $TM = \bigsqcup_{m \in M} T_xM$, is a *disjoint union* of tangent spaces T_xM to M for all points $x \in M$. From dynamical perspective, the most important quantity in the tangent bundle concept is the smooth map $v : M \to TM$, which is an inverse to the projection π, i.e, $\pi \circ v = \mathrm{Id}_M$, $\pi(v(x)) = x$. It is called the *velocity vector–field*. Its graph $(x, v(x))$ represents the *cross–section* of the tangent bundle TM. This explains the dynamical term *velocity phase–space*, given to the tangent bundle TM of the manifold M.

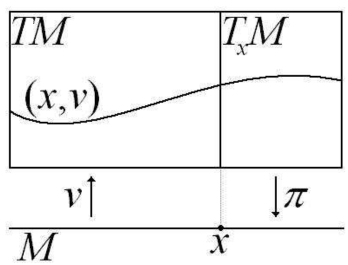


Figure 2: A sketch of a tangent bundle TM of a smooth manifold M.

If $[a, b]$ is a closed interval, a C^0−map $\gamma : [a, b] \to M$ is said to be *differentiable* at the endpoint a if there is a chart (U, ϕ) at $\gamma(a)$ such that the following limit exists and is finite:

$$\frac{d}{dt}(\phi \circ \gamma)(a) \equiv (\phi \circ \gamma)'(a) = \lim_{t \to a} \frac{(\phi \circ \gamma)(t) - (\phi \circ \gamma)(a)}{t - a}. \tag{1}$$

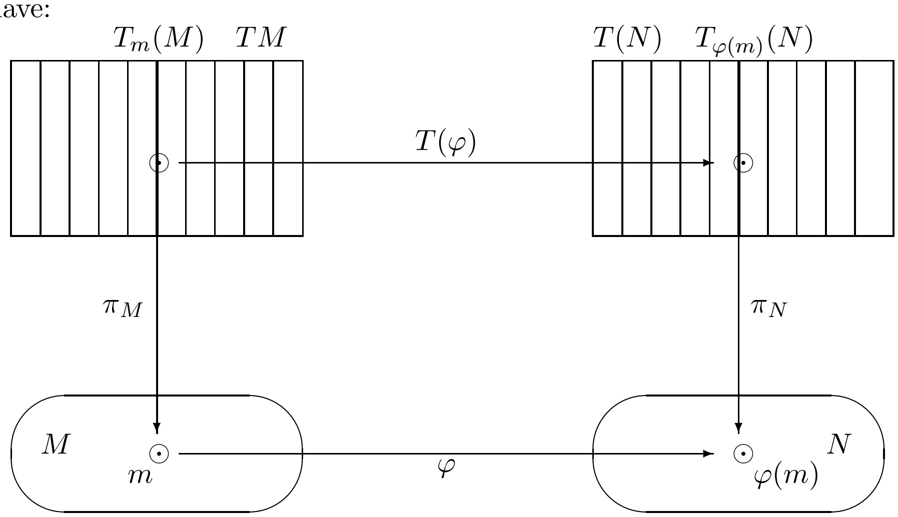
Generalizing (1), we get the notion of the *curve on a manifold*. For a smooth manifold M and a point $m \in M$ a curve at m is a C^0−map $\gamma : I \to M$ from an interval $I \subset \mathbb{R}$ into M with $0 \in I$ and $\gamma(0) = m$.

Two curves γ_1 and γ_2 passing though a point $m \in U$ are *tangent at* m with respect to the chart (U, ϕ) if $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. Thus, two curves are tangent if they have identical tangent vectors (same direction and speed) in a local chart on a manifold.

For a smooth manifold M and a point $m \in M$, the *tangent space* T_mM to M at m is the *set of equivalence classes* of curves at m:

$$T_mM = \{[\gamma]_m : \gamma \text{ is a curve at a point } m \in M\}.$$

A C^k–map $\varphi : M \ni m \mapsto \varphi(m) \in N$ between two manifolds M and N induces a linear map $T_m\varphi : T_mM \to T_{\varphi(m)}N$ for each point $m \in M$, called a *tangent map*, if we have:



i.e., the following diagram commutes:

$$\begin{tikzcd}
T_mM \arrow[r, "T_m\varphi"] \arrow[d, "\pi_M"'] & T_{\varphi(m)}N \arrow[d, "\pi_N"] \\
M \ni m \arrow[r, "\varphi"'] & \varphi(m) \in N
\end{tikzcd}$$

with the *natural projection* $\pi_M : TM \to M$, given by $\pi_M(T_mM) = m$, that takes a tangent vector v to the point $m \in M$ at which the vector v is attached i.e., $v \in T_mM$.

For an nD smooth manifold M, its nD *tangent bundle* TM is the disjoint union of all its tangent spaces T_mM at all points $m \in M$, $TM = \bigsqcup_{m\in M} T_mM$.

To define the smooth structure on TM, we need to specify how to construct local coordinates on TM. To do this, let $(x^1(m), ..., x^n(m))$ be local coordinates of a point m on M and let $(v^1(m), ..., v^n(m))$ be components of a tangent vector in this coordinate system. Then the $2n$ numbers $(x^1(m), ..., x^n(m), v^1(m), ..., v^n(m))$ give a *local coordinate system* on TM.

$TM = \bigsqcup_{m\in M} T_mM$ defines a family of vector spaces parameterized by M. The inverse image $\pi_M^{-1}(m)$ of a point $m \in M$ under the natural projection π_M is the tangent space T_mM. This space is called the *fibre* of the tangent bundle over the point $m \in M$.

A C^k-map $\varphi : M \to N$ between two manifolds M and N induces a linear *tangent map* $T\varphi : TM \to TN$ between their tangent bundles, i.e., the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "T\varphi"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

All tangent bundles and their tangent maps form the category $\mathcal{TB}$. The category $\mathcal{TB}$ is the natural framework for *Lagrangian dynamics*.

Now, we can formulate the *global version of the chain rule*. If $\varphi : M \to N$ and $\psi : N \to P$ are two smooth maps, then we have $T(\psi \circ \varphi) = T\psi \circ T\varphi$. In other words, we have a functor $T : \mathcal{M} \Rightarrow \mathcal{TB}$ from the category $\mathcal{M}$ of smooth manifolds to the category $\mathcal{TB}$ of their tangent bundles:

$$\begin{tikzcd} & M \arrow[dl, "\varphi"'] \arrow[dr, "(\psi\circ\varphi)"] & & & & TM \arrow[dl, "T\varphi"'] \arrow[dr, "T(\psi\circ\varphi)"] & \\ N \arrow[rr, "\psi"'] & & P & \overset{T}{\Longrightarrow} & TN \arrow[rr, "T\psi"'] & & TP \end{tikzcd}$$

3.5 Cotangent Bundle and Hamiltonian Dynamics

The cotangent bundle of a smooth $n-$manifold is the place is where 1–forms live, and is itself a smooth $2n-$manifold. Covector–fields (1–forms) are cross-sections of the cotangent bundle. The *Hamiltonian* is a natural energy function on the cotangent bundle (see [8, 9]).

A *dual* notion to the tangent space T_mM to a smooth manifold M at a point m is its *cotangent space* T_m^*M at the same point m. Similarly to the tangent bundle, for a smooth manifold M of dimension n, its *cotangent bundle* T^*M is the disjoint union of all its cotangent spaces T_m^*M at all points $m \in M$, i.e., $T^*M = \bigsqcup_{m\in M} T_m^*M$. Therefore, the cotangent bundle of an $n-$manifold M is the vector bundle $T^*M = (TM)^*$, the (real) dual of the tangent bundle TM.

If M is an $n-$manifold, then T^*M is a $2n-$manifold. To define the smooth structure on T^*M, we need to specify how to construct local coordinates on T^*M. To do this, let $(x^1(m), ..., x^n(m))$ be local coordinates of a point m on M and let $(p_1(m), ..., p_n(m))$ be components of a covector in this coordinate system. Then the $2n$ numbers $(x^1(m), ..., x^n(m), p_1(m), ..., p_n(m))$ give a local coordinate system on T^*M. This is the basic idea one uses to prove that indeed T^*M is a $2n-$manifold.

$T^*M = \bigsqcup_{m \in M} T_m^*M$ defines a family of vector spaces parameterized by M, with the *conatural projection*, $\pi_M^* : T^*M \to M$, given by $\pi_M^*(T_m^*M) = m$, that takes a covector p to the point $m \in M$ at which the covector p is attached i.e., $p \in T_m^*M$. The inverse image $\pi_M^{-1}(m)$ of a point $m \in M$ under the conatural projection π_M^* is the cotangent space T_m^*M. This space is called the *fibre* of the cotangent bundle over the point $m \in M$.

In a similar way, a C^k–map $\varphi : M \to N$ between two manifolds M and N induces a linear *cotangent map* $T^*\varphi : T^*M \to T^*N$ between their cotangent bundles, i.e., the following diagram commutes:

$$\begin{tikzcd}
T^*M \arrow[r, "T^*\varphi"] \arrow[d, "\pi_M^*"'] & T^*N \arrow[d, "\pi_N^*"] \\
M \arrow[r, "\varphi"'] & N
\end{tikzcd}$$

All cotangent bundles and their cotangent maps form the category $\mathcal{T}^*\mathcal{B}$. The category $\mathcal{T}^*\mathcal{B}$ is the natural stage for *Hamiltonian dynamics*.

Now, we can formulate the *dual version of the global chain rule*. If $\varphi : M \to N$ and $\psi : N \to P$ are two smooth maps, then we have $T^*(\psi \circ \varphi) = T^*\psi \circ T^*\varphi$. In other words, we have a cofunctor $T^* : \mathcal{M} \Rightarrow \mathcal{T}^*\mathcal{B}$ from the category $\mathcal{M}$ of smooth manifolds to the category $\mathcal{T}^*\mathcal{B}$ of their cotangent bundles:

$$\begin{tikzcd}
 & M \arrow[dl, "\varphi"'] \arrow[dr, "(\psi \circ \varphi)"] & \\
N \arrow[rr, "\psi"'] & & P
\end{tikzcd}
\quad \overset{T^*}{\Longrightarrow} \quad
\begin{tikzcd}
 & T^*M & \\
T^*N \arrow[ur, "T^*\varphi"] & & T^*P \arrow[ul, "T^*(\psi \circ \varphi)"'] \arrow[ll, "T^*\psi"]
\end{tikzcd}$$

4 Lie Groups

In this section we present the basics of *classical theory of Lie groups* and their Lie algebras, as developed mainly by Sophus Lie, Elie Cartan, Felix Klein, Wilhelm Killing and Hermann Weyl. For more comprehensive treatment see e.g., [21, 22, 23, 24, 25].

In the middle of the 19th Century S. Lie made a far reaching discovery that techniques designed to solve particular unrelated types of ODEs, such as separable, homogeneous and exact equations, were in fact all special cases of a general form of integration procedure based on the invariance of the differential equation under a

continuous group of symmetries. Roughly speaking a symmetry group of a system of differential equations is a group that transforms solutions of the system to other solutions. Once the symmetry group has been identified a number of techniques to solve and classify these differential equations becomes possible. In the classical framework of Lie, these groups were local groups and arose locally as groups of transformations on some Euclidean space. The passage from the local Lie group to the present day definition using manifolds was accomplished by E. Cartan at the end of the 19th Century, whose work is a striking synthesis of Lie theory, classical geometry, differential geometry and topology.

These continuous groups, which originally appeared as symmetry groups of differential equations, have over the years had a profound impact on diverse areas such as algebraic topology, differential geometry, numerical analysis, control theory, classical mechanics, quantum mechanics etc. They are now universally known as Lie groups.

A Lie group is smooth manifold which also carries a group structure whose product and inversion operations are smooth as maps of manifolds. These objects arise naturally in describing physical symmetries.[6]

[6] Here are a few examples of Lie groups and their relations to other areas of mathematics and physics:

1. Euclidean space $\mathbb{R}^n$ is an Abelian Lie group (with ordinary vector addition as the group operation).
2. The group $GL_n(\mathbb{R})$ of invertible matrices (under matrix multiplication) is a Lie group of dimension n^2. It has a subgroup $SL_n(\mathbb{R})$ of matrices of determinant 1 which is also a Lie group.
3. The group $O_n(\mathbb{R})$ generated by all rotations and reflections of an nD vector space is a Lie group called the *orthogonal group*. It has a subgroup of elements of determinant 1, called the special orthogonal group $SO(n)$, which is the *group of rotations* in $\mathbb{R}^n$.
4. Spin groups are double covers of the special orthogonal groups (used e.g., for studying fermions in quantum field theory).
5. The group $Sp_{2n}(\mathbb{R})$ of all matrices preserving a symplectic form is a Lie group called the *symplectic group*.
6. The Lorentz group and the Poincaré group of isometries of space–time are Lie groups of dimensions 6 and 10 that are used in special relativity.
7. The Heisenberg group is a Lie group of dimension 3, used in quantum mechanics.
8. The unitary group $U(n)$ is a compact group of dimension n^2 consisting of unitary matrices. It has a subgroup of elements of determinant 1, called the special unitary group $SU(n)$.
9. The group $U(1) \times SU(2) \times SU(3)$ is a Lie group of dimension $1 + 3 + 8 = 12$ that is the *gauge group* of the *Standard Model* of elementary particles, whose dimension corresponds to: 1 photon + 3 vector bosons + 8 gluons.

A Lie group is a group whose elements can be continuously parametrized by real numbers, such as the *rotation group* $SO(3)$, which can be parametrized by the *Euler angles*. More formally, a Lie group is an analytic real or complex manifold that is also a group, such that the group operations multiplication and inversion are analytic maps. Lie groups are important in mathematical analysis, physics and geometry because they serve to describe the symmetry of analytical structures. They were introduced by *Sophus Lie* in 1870 in order to study symmetries of differential equations.

While the Euclidean space $\mathbb{R}^n$ is a *real Lie group* (with ordinary vector addition as the group operation), more typical examples are given by matrix Lie groups, i.e., groups of invertible matrices (under matrix multiplication). For instance, the group $SO(3)$ of all rotations in $\mathbb{R}^3$ is a matrix Lie group.

One classifies Lie groups regarding their *algebraic properties*[7] (simple, semisimple, solvable, nilpotent, Abelian), their *connectedness* (connected or simply connected) and their *compactness*.[8]

To every Lie group, we can associate a *Lie algebra* which completely captures the local structure of the group (at least if the Lie group is connected).[9]

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[7]If G and H are Lie groups (both real or both complex), then a *Lie–group–homomorphism* $f : G \to H$ is a *group homomorphism* which is also an *analytic map* (one can show that it is equivalent to require only that f be continuous). The composition of two such homomorphisms is again a homomorphism, and the class of all (real or complex) Lie groups, together with these morphisms, forms a *category*. The two Lie groups are called *isomorphic* iff there exists a bijective homomorphism between them whose inverse is also a homomorphism. Isomorphic Lie groups do not need to be distinguished for all practical purposes; they only differ in the notation of their elements.

[8]An $n-$torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ (as defined above) is an example of a *compact Abelian Lie group*. This follows from the fact that the unit circle S^1 is a compact Abelian Lie group (when identified with the unit complex numbers with multiplication). Group multiplication on T^n is then defined by coordinate–wise multiplication.

Toroidal groups play an important part in the theory of compact Lie groups. This is due in part to the fact that in any compact Lie group one can always find a maximal torus; that is, a closed subgroup which is a torus of the largest possible dimension.

[9]Conventionally, one can regard any field X of tangent vectors on a Lie group as a partial differential operator, denoting by Xf the *Lie derivative* (the *directional derivative*) of the scalar field f in the direction of X. Then a vector–field on a Lie group G is said to be left–invariant if it commutes with left translation, which means the following. Define $L_g[f](x) = f(gx)$ for any analytic function $f : G \to \mathbb{R}$ and all $g, x \in G$. Then the vector–field X is left–invariant iff $XL_g = L_gX$ for all $g \in G$. Similarly, instead of $\mathbb{R}$, we can use $\mathbb{C}$. The set of all vector–fields on an analytic manifold is a *Lie algebra* over $\mathbb{R}$ (or $\mathbb{C}$).

On a Lie group G, the left–invariant vector–fields form a subalgebra, the Lie algebra $\mathfrak{g}$ associated with G. This Lie algebra is finite–dimensional (it has the same dimension as the manifold G) which makes it susceptible to classification attempts. By classifying $\mathfrak{g}$, one can also get a handle on the group G. The representation theory of simple Lie groups is the best and most important example.

Every element v of the tangent space T_e at the identity element e of G determines a unique

4.1 Definition of a Lie Group

A *Lie group* is a smooth (Banach) manifold M that has at the same time a group $G-$structure consistent with its manifold $M-$structure in the sense that *group multiplication* $\mu : G \times G \to G, \quad (g, h) \mapsto gh$ and the *group inversion* $\nu : G \to G, \quad g \mapsto g^{-1}$ are C^k-maps. A point $e \in G$ is called the *group identity element* (see e.g., [21, 22, 1, 3]).

For example, any nD Banach vector space V is an Abelian Lie group with group operations $\mu : V \times V \to V$, $\mu(x, y) = x + y$, and $\nu : V \to V$, $\nu(x) = -x$. The identity is just the zero vector. We call such a Lie group a *vector group*.

Let G and H be two Lie groups. A map $G \to H$ is said to be a *morphism* of Lie groups (or their *smooth homomorphism*) if it is their homomorphism as abstract groups and their smooth map as manifolds.

Similarly, a group G which is at the same time a topological space is said to be a *topological group* if both maps (μ, ν) are continuous, i.e., C^0-maps for it. The homomorphism $G \to H$ of topological groups is said to be continuous if it is a continuous map.

A topological group (as well as a smooth manifold) is not necessarily Hausdorff.

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left–invariant vector–field whose value at the element g of G is denoted by gv; the vector space underlying the Lie algebra $\mathfrak{g}$ may therefore be identified with T_e.

Every vector–field v in the Lie algebra $\mathfrak{g}$ determines a function $c : \mathbb{R} \to G$ whose derivative everywhere is given by the corresponding left–invariant vector–field: $c'(t) = TL_{c(t)}v$ and which has the property: $c(s+t) = c(s)c(t)$, (for all s and t) (the operation on the r.h.s. is the group multiplication in G). The formal similarity of this formula with the one valid for the elementary exponential function justifies the definition: $\exp(v) = c(1)$. This is called the *exponential map*, and it maps the Lie algebra $\mathfrak{g}$ into the Lie group G. It provides a *diffeomorphism* between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of e in G. This exponential map is a generalization of the exponential function for real numbers (since $\mathbb{R}$ is the Lie algebra of the Lie group of positive real numbers with multiplication), for complex numbers (since $\mathbb{C}$ is the Lie algebra of the Lie group of non–zero complex numbers with multiplication) and for matrices (since $M(n, \mathbb{R})$ with the regular commutator is the Lie algebra of the Lie group $GL(n, \mathbb{R})$ of all invertible matrices). As the exponential map is surjective on some neighborhood N of e, it is common to call elements of the Lie algebra *infinitesimal generators* of the group G.

The exponential map and the Lie algebra determine the local group structure of every connected Lie group, because of the *Baker–Campbell–Hausdorff formula*: there exists a neighborhood U of the zero element of the Lie algebra $\mathfrak{g}$, such that for $u, v \in U$ we have

$$\exp(u)\exp(v) = \exp(u + v + 1/2[u, v] + 1/12[[u, v], v] - 1/12[[u, v], u] - ...),$$

where the omitted terms are known and involve *Lie brackets* of four or more elements. In case u and v commute, this formula reduces to the familiar *exponential law*:

$$\exp(u)\exp(v) = \exp(u + v).$$

Every homomorphism $f : G \to H$ of Lie groups induces a homomorphism between the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The association $G \Longrightarrow \mathfrak{g}$ is called the *Lie Functor*.

A topological group G is Hausdorff iff its identity is closed. As a corollary we have that every Lie group is a Hausdorff topological group.

For every g in a Lie group G, the two maps,

$$\begin{aligned} L_g &: G \to G, \qquad h \mapsto gh, \\ R_h &: G \to G, \qquad g \mapsto gh, \end{aligned}$$

are called *left* and *right translation* maps. Since $L_g \circ L_h = L_{gh}$, and $R_g \circ R_h = R_{gh}$, it follows that $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$, so both L_g and R_g are diffeomorphisms. Moreover $L_g \circ R_h = R_h \circ L_g$, i.e., left and right translation commute.

A vector–field X on G is called *left–invariant vector–field* if for every $g \in G$, $L_g^* X = X$, that is, if $(T_h L_g) X(h) = X(gh)$ for all $h \in G$, i.e., the following diagram commutes:

$$\begin{tikzcd} TG \arrow[r, "TL_g"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_g"'] & G \arrow[u, "X"'] \end{tikzcd}$$

A *Riemannian metric* on a Lie group G is called left-invariant if it is preserved by all left translations L_g, i.e., if the derivative of left translation carries every vector to a vector of the same length. Similarly, a vector field X on G is called left–invariant if (for every $g \in G$) $L_g^* X = X$.

4.2 Lie Algebra

An *algebra* A is a vector space with a product. The product must have the property that

$$a(uv) = (au)v = u(av),$$

for every $a \in \mathbb{R}$ and $u, v \in A$. A map $\phi : A \to A'$ between algebras is called an *algebra homomorphism* if $\phi(u \cdot v) = \phi(u) \cdot \phi(v)$. A vector subspace $\mathfrak{I}$ of an algebra A is called a *left ideal* (resp. *right ideal*) if it is closed under algebra multiplication and if $u \in A$ and $i \in \mathfrak{I}$ implies that $ui \in \mathfrak{I}$ (resp. $iu \in \mathfrak{I}$). A subspace $\mathfrak{I}$ is said to be a *two–sided ideal* if it is both a left and right ideal. An ideal may not be an algebra itself, but the quotient of an algebra by a two–sided ideal inherits an algebra structure from A.

A *Lie algebra* is an algebra A where the multiplication, i.e., the *Lie bracket* $(u, v) \mapsto [u, v]$, has the following properties:

LA 1. $[u, u] = 0$ for every $u \in A$, and

LA 2. $[u, [v, w]] + [w, [u, v]] + [v, w, u]] = 0$ for all $u, v, w \in A$.

The condition LA 2 is usually called *Jacobi identity.* A subspace $E \subset A$ of a Lie algebra is called a *Lie subalgebra* if $[u, v] \in E$ for every $u, v \in E$. A map $\phi : A \to A'$ between Lie algebras is called a *Lie algebra homomorphism* if $\phi([u, v]) = [\phi(u), \phi(v)]$ for each $u, v \in A$.

All Lie algebras (over a given field $\mathbb{K}$) and all smooth homomorphisms between them form the category $\mathcal{LAL}$, which is itself a complete subcategory of the category $\mathcal{AL}$ of all algebras and their homomorphisms.

Let $\mathcal{X}_L(G)$ denote the set of left–invariant vector–fields on G; it is a Lie subalgebra of $\mathcal{X}(G)$, the set of all vector–fields on G, since $L_g^*[X, Y] = [L_g^* X, L_g^* Y] = [X, Y]$, so the Lie bracket $[X, Y] \in \mathcal{X}_L(G)$.

Let e be the identity element of G. Then for each ξ on the tangent space T_eG we define a vector–field X_ξ on G by $X_\xi(g) = T_eL_g(\xi)$. $\mathcal{X}_L(G)$ and T_eG are isomorphic as vector spaces. Define the Lie bracket on T_eG by $[\xi, \eta] = [X_\xi, X_\eta](e)$ for all $\xi, \eta \in T_eG$. This makes T_eG into a Lie algebra. Also, by construction, we have $[X_\xi, X_\eta] = X_{[\xi,\eta]}$; this defines a bracket in T_eG via *left extension.* The vector space T_eG with the above algebra structure is called the Lie algebra of the Lie group G and is denoted $\mathfrak{g}$.

For example, let V be a nD vector space. Then $T_eV \simeq V$ and the left–invariant vector–field defined by $\xi \in T_eV$ is the constant vector–field $X_\xi(\eta) = \xi$, for all $\eta \in V$. The Lie algebra of V is V itself.

Since any two elements of an Abelian Lie group G commute, it follows that all adjoint operators Ad_g, $g \in G$, equal the identity. Therefore, the Lie algebra g is Abelian; that is, $[\xi, \eta] = 0$ for all $\xi, \eta \in \mathfrak{g}$.

For example, $G = SO(3)$ is the group of rotations of 3D Euclidean space, i.e. the configuration space of a rigid body fixed at a point. A motion of the body is then described by a curve $g = g(t)$ in the group $SO(3)$. Its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ is the 3D vector space of angular velocities of all possible rotations. The commutator in this algebra is the usual vector (cross) product (see, e.g. [1, 3, 9]).

A rotation velocity $\dot{g}$ of the rigid body (fixed at a point) is a tangent vector to the Lie group $G = SO(3)$ at the point $g \in G$. To get the angular velocity, we must carry this vector to the tangent space TG_e of the group at the identity, i.e. to its Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$. This can be done in two ways: by left and right translation, L_g and R_g. As a result, we obtain two different vector fields in the Lie algebra $\mathfrak{so}(3)$:

$$\omega_c = L_{g^{-1}*}\dot{g} \in \mathfrak{so}(3) \qquad \text{and} \qquad \omega_x = R_{g^{-1}*}\dot{g} \in \mathfrak{so}(3),$$

which are called the 'angular velocity in the body' and the 'angular velocity in space,' respectively.

The dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ is the space of angular momenta $\boldsymbol{\pi}$. The kinetic energy T of a body is determined by the vector field of angular

velocity in the body and does not depend on the position of the body in space. Therefore, kinetic energy gives a left-invariant Riemannian metric on the rotation group $G = SO(3)$.

4.3 One-Parameter Subgroup

Let X_ξ be a left–invariant vector–field on G corresponding to ξ in $\mathfrak{g}$. Then there is a unique integral curve $\gamma_\xi : \mathbb{R} \to G$ of X_ξ starting at e, i.e., (see, e.g. [8, 9])

$$\dot{\gamma}_\xi(t) = X_\xi\left(\gamma_\xi(t)\right), \qquad \gamma_\xi(0) = e$$

$\gamma_\xi(t)$ is a smooth *one–parameter subgroup* of G, i.e., $\gamma_\xi(t+s) = \gamma_\xi(t) \cdot \gamma_\xi(s)$, since, as functions of t both sides equal $\gamma_\xi(s)$ at $t = 0$ and both satisfy differential equation $\dot{\gamma}(t) = X_\xi\left(\gamma_\xi(t)\right)$ by left invariance of X_ξ, so they are equal. Left invariance can be also used to show that $\gamma_\xi(t)$ is defined for all $t \in \mathbb{R}$. Moreover, if $\phi : \mathbb{R} \to G$ is a one–parameter subgroup of G, i.e., a *smooth homomorphism* of the additive group $\mathbb{R}$ into G, then $\phi = \gamma_\xi$ with $\xi = \dot{\phi}(0)$, since taking derivative at $s = 0$ in the relation

$$\phi(t+s) = \phi(t) \cdot \phi(s) \qquad \text{gives} \qquad \dot{\phi}(t) = X_{\dot{\phi}(0)}\left(\phi(t)\right),$$

so $\phi = \gamma_\xi$ since both equal e at $t = 0$. Therefore, all one–parameter subgroups of G are of the form $\gamma_\xi(t)$ for some $\xi \in \mathfrak{g}$.

4.4 Exponential Map

The map $\exp : \mathfrak{g} \to G$, given by (see, e.g. [3, 8, 9]):

$$\exp(\xi) = \gamma_\xi(1), \qquad \exp(0) = e$$

is called the *exponential map* of the Lie algebra $\mathfrak{g}$ of G into G. exp is a C^k–-map, similar to the projection π of tangent and cotangent bundles; exp is locally a diffeomorphism from a neighborhood of zero in $\mathfrak{g}$ onto a neighborhood of e in G; if $f : G \to H$ is a smooth homomorphism of Lie groups, then

$$f \circ \exp_G = \exp_H \circ T_e f\,.$$

Also, in this case

$$\exp(s\xi) = \gamma_\xi(s).$$

Indeed, for fixed $s \in \mathbb{R}$, the curve $t \mapsto \gamma_\xi(ts)$, which at $t = 0$ passes through e, satisfies the differential equation

$$\frac{d}{dt}\gamma_\xi(ts) = sX_\xi\left(\gamma_\xi(ts)\right) = X_{s\xi}\left(\gamma_\xi(ts)\right).$$

Since $\gamma_{s\xi}(t)$ satisfies the same differential equation and passes through e at $t = 0$, it follows that $\gamma_{s\xi}(t) = \gamma_{\xi}(st)$. Putting $t = 1$ induces $\exp(s\xi) = \gamma_{\xi}(s)$.

Hence exp maps the line $s\xi$ in $\mathfrak{g}$ onto the one–parameter subgroup $\gamma_{\xi}(s)$ of G, which is tangent to ξ at e. It follows from left invariance that the flow F_t^{ξ} of X satisfies $F_t^{\xi}(g) = g\exp(s\xi)$.

Globally, the exponential map exp is a natural operation, i.e., for any morphism $\varphi : G \to H$ of Lie groups G and H and a Lie functor $\mathcal{F}$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}(G) \arrow[r, "\mathcal{F}(\varphi)"] \arrow[d, "\exp"'] & \mathcal{F}(H) \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

Let G_1 and G_2 be Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Then $G_1 \times G_2$ is a Lie group with Lie algebra $\mathfrak{g}_1 \times \mathfrak{g}_2$, and the exponential map is given by:

$$\exp : \mathfrak{g}_1 \times \mathfrak{g}_2 \to G_1 \times G_2, \qquad (\xi_1, \xi_2) \mapsto (\exp_1(\xi_1), \exp_2(\xi_2)) .$$

For example, in case of a nD vector space, or infinite–dimensional Banach space, the exponential map is the identity.

The unit circle in the complex plane $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is an Abelian Lie group under multiplication. The tangent space T_eS^1 is the imaginary axis, and we identify $\mathbb{R}$ with T_eS^1 by $t \mapsto 2\pi it$. With this identification, the exponential map $\exp : \mathbb{R} \to S^1$ is given by $\exp(t) = \mathrm{e}^{2\pi it}$.

The nD torus $T^n = S^1 \times \cdots \times S^1$ (n times) is an Abelian Lie group. The exponential map $\exp : \mathbb{R}^n \to T^n$ is given by

$$\exp(t_1, ..., t_n) = (\mathrm{e}^{2\pi it_1}, ..., \mathrm{e}^{2\pi it_n}).$$

Since $S^1 = \mathbb{R}/\mathbb{Z}$, it follows that $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the projection $\mathbb{R}^n \to T^n$ being given by the exp map.

4.5 Adjoint Representation

For every $g \in G$, the map (see, e.g. [1, 3, 8, 9]):

$$Ad_g = T_e\left(R_{g^{-1}} \circ L_g\right) : \mathfrak{g} \to \mathfrak{g}$$

is called the *adjoint map*, or *adjoint operator* associated with g.

For each $\xi \in \mathfrak{g}$ and $g \in G$ we have

$$\exp\left(Ad_g\xi\right) = g\left(\exp\xi\right)g^{-1}.$$

The relation between the adjoint map and the Lie bracket is the following: For all $\xi, \eta \in \mathfrak{g}$ we have

$$\frac{d}{dt}\bigg|_{t=0} Ad_{\exp(t\xi)}\eta = [\xi, \eta].$$

Left and right translations induce operators on the cotangent space T^*G_g dual to L_{g*} and R_{g*}, denoted by (for every $h \in G$):

$$L_g^* : T^*G_{gh} \to T^*G_h, \qquad R_g^* : T^*G_{hg} \to T^*G_h.$$

The transpose operators $Ad_g^* : \mathfrak{g} \to \mathfrak{g}$ satisfy the relations $Ad_{gh}^* = Ad_h^* Ad_g^*$ (for every $g, h \in G$) and constitute the *co-adjoint representation* of the Lie group G. The co-adjoint representation plays an important role in all questions related to (left) invariant metrics on the Lie group. According to A. Kirillov, the orbit of any vector field X in a Lie algebra $\mathfrak{g}$ in a co-adjoint representation Ad_g^* is itself a symplectic manifold and therefore a phase space for a Hamiltonian mechanical system.

A Lie subgroup H of G is a subgroup H of G which is also a submanifold of G. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and moreover $\mathfrak{h} = \{\xi \in \mathfrak{g} | \exp(t\xi) \in H$, for all $t \in \mathbb{R}\}$.

One can characterize *Lebesgue measure* up to a multiplicative constant on $\mathbb{R}^n$ by its invariance under translations. Similarly, on a locally compact group there is a unique (up to a nonzero multiplicative constant) left–invariant measure, called *Haar measure*. For Lie groups the existence of such measures is especially simple: Let G be a Lie group. Then there is a volume form $Ub5$, unique up to nonzero multiplicative constants, that is left–invariant. If G is compact, $Ub5$ is right invariant as well.

4.6 Actions of Lie Groups on Smooth Manifolds

Let M be a smooth manifold. An *action of a Lie group* G (with the unit element e) on M is a smooth map $\phi : G \times M \to M$, such that for all $x \in M$ and $g, h \in G$, (i) $\phi(e, x) = x$ and (ii) $\phi(g, \phi(h, x)) = \phi(gh, x)$. In other words, letting $\phi_g : x \in M \mapsto \phi_g(x) = \phi(g, x) \in M$, we have (i') $\phi_e = id_M$ and (ii') $\phi_g \circ \phi_h = \phi_{gh}$. ϕ_g is a diffeomorphism, since $(\phi_g)^{-1} = \phi_{g^{-1}}$. We say that the map $g \in G \mapsto \phi_g \in Diff(M)$ is a homomorphism of G into the group of diffeomorphisms of M. In case that M is a vector space and each ϕ_g is a linear operator, the function of G on M is called a representation of G on M (see, e.g. [1, 3, 8, 9]).

An action ϕ of G on M is said to be *transitive group action*, if for every $x, y \in M$, there is $g \in G$ such that $\phi(g, x) = y$; *effective group action*, if $\phi_g = id_M$ implies $g = e$, that is $g \mapsto \phi_g$ is 1–1; and *free group action*, if for each $x \in M$, $g \mapsto \phi_g(x)$ is 1–1.

For example,

1. $G = \mathbb{R}$ acts on $M = \mathbb{R}$ by translations; explicitly,

$$\phi : G \times M \to M, \qquad \phi(s, x) = x + s.$$

   Then for $x \in \mathbb{R}$, $O_x = \mathbb{R}$. Hence M/G is a single point, and the action is transitive and free.

2. A complete flow ϕ_t of a vector–field X on M gives an action of $\mathbb{R}$ on M, namely

$$(t, x) \in \mathbb{R} \times M \mapsto \phi_t(x) \in M.$$

3. Left translation $L_g : G \to G$ defines an effective action of G on itself. It is also transitive.

4. The coadjoint action of G on $\mathfrak{g}^*$ is given by

$$Ad^* : (g, \alpha) \in G \times \mathfrak{g}^* \mapsto Ad^*_{g^{-1}}(\alpha) = \left(T_e(R_{g^{-1}} \circ L_g)\right)^* \alpha \in \mathfrak{g}^*.$$

Let ϕ be an action of G on M. For $x \in M$ the *orbit* of x is defined by

$$O_x = \{\phi_g(x) | g \in G\} \subset M$$

and the *isotropy group* of ϕ at x is given by

$$G_x = \{g \in G | \phi(g, x) = x\} \subset G.$$

An action ϕ of G on a manifold M defines an equivalence relation on M by the relation belonging to the same orbit; explicitly, for $x, y \in M$, we write $x \sim y$ if there exists a $g \in G$ such that $\phi(g, x) = y$, that is, if $y \in O_x$. The set of all orbits M/G is called the *group orbit space* (see, e.g. [1, 3, 8, 9]).

For example, let $M = \mathbb{R}^2 \backslash \{0\}$, $G = SO(2)$, the group of rotations in plane, and the action of G on M given by

$$\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, (x, y) \right) \longmapsto (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

The action is always free and effective, and the orbits are concentric circles, thus the orbit space is $M/G \simeq \mathbb{R}^*_+$.

A crucial concept in mechanics is the *infinitesimal description of an action.* Let $\phi : G \times M \to M$ be an action of a Lie group G on a smooth manifold M. For each $\xi \in \mathfrak{g}$,

$$\phi_\xi : \mathbb{R} \times M \to M, \qquad \phi_\xi(t, x) = \phi\left(\exp(t\xi), x\right)$$

is an $\mathbb{R}-$-action on M. Therefore, $\phi_{\exp(t\xi)} : M \to M$ is a flow on M; the corresponding vector–field on M, given by

$$\xi_M(x) = \frac{d}{dt}\bigg|_{t=0} \phi_{\exp(t\xi)}(x)$$

is called the infinitesimal generator of the action, corresponding to ξ in $\mathfrak{g}$.

The tangent space at x to an orbit O_x is given by

$$T_x O_x = \{\xi_M(x) | \xi \in \mathfrak{g}\}.$$

Let $\phi : G \times M \to M$ be a smooth $G--$action. For all $g \in G$, all $\xi, \eta \in \mathfrak{g}$ and all $\alpha, \beta \in \mathbb{R}$, we have:

$(Ad_g \xi)_M = \phi^*_{g^{-1}} \xi_M$, $[\xi_M, \eta_M] = -[\xi, \eta]_M$, and $(\alpha\xi + \beta\eta)_M = \alpha\xi_M + \beta\eta_M$.

Let M be a smooth manifold, G a Lie group and $\phi : G \times M \to M$ a $G-$action on M. We say that a smooth map $f : M \to M$ is with respect to this action if for all $g \in G$,

$$f \circ \phi_g = \phi_g \circ f.$$

Let $f : M \to M$ be an equivariant smooth map. Then for any $\xi \in \mathfrak{g}$ we have

$$Tf \circ \xi_M = \xi_M \circ f.$$

4.7 Basic Tables of Lie Groups and Their Lie Algebras

One classifies Lie groups regarding their algebraic properties (simple, semisimple, solvable, nilpotent, Abelian), their connectedness (connected or simply connected) and their compactness (see Tables A.1–A.3). This is the content of the *Hilbert 5th problem.*

Some real Lie groups and their Lie algebras:

Lie group	Description	Remarks	Lie algb.	Description	dim $/\mathbb{R}$
$\mathbb{R}^n$	Euclidean space with addition	Abelian, simply connected, not compact	$\mathbb{R}^n$	the Lie bracket is zero	n
$\mathbb{R}^\times$	nonzero real numbers with multiplication	Abelian, not connected, not compact	$\mathbb{R}$	the Lie bracket is zero	1
$\mathbb{R}^{>0}$	positive real numbers with multiplication	Abelian, simply connected, not compact	$\mathbb{R}$	the Lie bracket is zero	1
$S^1 = \mathbb{R}/\mathbb{Z}$	complex numbers of absolute value 1, with multiplication	Abelian, connected, not simply connected, compact	$\mathbb{R}$	the Lie bracket is zero	1
$\mathbb{H}^\times$	non–zero quaternions with multiplication	simply connected, not compact	$\mathbb{H}$	quaternions, with Lie bracket the commutator	4
S^3	quaternions of absolute value 1, with multiplication; a $3-$sphere	simply connected, compact, simple and semi–simple, isomorphic to $SU(2)$, $SO(3)$ and to $Spin(3)$	$\mathbb{R}^3$	real $3-$vectors, with Lie bracket the cross product; isomorphic to $\mathfrak{su}(2)$ and to $\mathfrak{so}(3)$	3
$GL(n,\mathbb{R})$	general linear group: invertible $n-$by-n real matrices	not connected, not compact	$\mathrm{M}(n,\mathbb{R})$	$n-$by-n matrices, with Lie bracket the commutator	n^2
$GL^+(n,\mathbb{R})$	$n-$by-n real matrices with positive determinant	simply connected, not compact	$\mathrm{M}(n,\mathbb{R})$	$n-$by-n matrices, with Lie bracket the commutator	n^2

Classical real Lie groups and their Lie algebras:

Lie group	**Description**	**Remarks**	**Lie algb.**	**Description**	**dim /$\mathbb{R}$**
$SL(n,\mathbb{R})$	special linear group: real matrices with determinant 1	simply connected, not compact if $n > 1$	$\mathfrak{sl}(n,\mathbb{R})$	square matrices with trace 0, with Lie bracket the commutator	n^2-1
$O(n,\mathbb{R})$	orthogonal group: real orthogonal matrices	not connected, compact	$\mathfrak{so}(n,\mathbb{R})$	skew–symmetric square real matrices, with Lie bracket the commutator; $\mathfrak{so}(3,\mathbb{R})$ is isomorphic to $\mathfrak{su}(2)$ and to $\mathbb{R}^3$ with the cross product	$n(n-1)/2$
$SO(n,\mathbb{R})$	special orthogonal group: real orthogonal matrices with determinant 1	connected, compact, for $n \geq 2$: not simply connected, for $n = 3$ and $n \geq 5$: simple and semisimple	$\mathfrak{so}(n,\mathbb{R})$	skew–symmetric square real matrices, with Lie bracket the commutator	$n(n-1)/2$
$Spin(n)$	spinor group	simply connected, compact, for $n = 3$ and $n \geq 5$: simple and semisimple	$\mathfrak{so}(n,\mathbb{R})$	skew–symmetric square real matrices, with Lie bracket the commutator	$n(n-1)/2$
$U(n)$	unitary group: complex unitary $n-$by-n matrices	isomorphic to S^1 for $n = 1$, not simply connected, compact	$\mathfrak{u}(n)$	square complex matrices A satisfying $A = -A^*$, with Lie bracket the commutator	n^2
$SU(n)$	special unitary group: complex unitary $n-$by-n matrices with determinant 1	simply connected, compact, for $n \geq 2$: simple and semisimple	$\mathfrak{su}(n)$	square complex matrices A with trace 0 satisfying $A = -A^*$, with Lie bracket the commutator	n^2-1

Basic complex Lie groups and their Lie algebras:[10]

Lie group	**Description**	**Remarks**	**Lie algb.**	**Description**	**dim /$\mathbb{C}$**
$\mathbb{C}^n$	group operation is addition	Abelian, simply connected, not compact	$\mathbb{C}^n$	the Lie bracket is zero	n
$\mathbb{C}^\times$	nonzero complex numbers with multiplication	Abelian, not simply connected, not compact	$\mathbb{C}$	the Lie bracket is zero	1
$GL(n,\mathbb{C})$	general linear group: invertible n−by-n complex matrices	simply connected, not compact, for $n = 1$: isomorphic to $\mathbb{C}^\times$	$M(n,\mathbb{C})$	n−by-n matrices, with Lie bracket the commutator	n^2
$SL(n,\mathbb{C})$	special linear group: complex matrices with determinant 1	simple, semisimple, simply connected, for $n \geq 2$: not compact	$\mathfrak{sl}(n,\mathbb{C})$	square matrices with trace 0, with Lie bracket the commutator	n^2-1
$O(n,\mathbb{C})$	orthogonal group: complex orthogonal matrices	not connected, for $n \geq 2$: not compact	$\mathfrak{so}(n,\mathbb{C})$	skew–symmetric square complex matrices, with Lie bracket the commutator	$n(n-1)/2$
$SO(n,\mathbb{C})$	special orthogonal group: complex orthogonal matrices with determinant 1	for $n \geq 2$: not compact, not simply connected, for $n = 3$ and $n \geq 5$: simple and semisimple	$\mathfrak{so}(n,\mathbb{C})$	skew–symmetric square complex matrices, with Lie bracket the commutator	$n(n-1)/2$

4.8 Representations of Lie groups

The idea of a *representation of a Lie group* plays an important role in the study of continuous symmetry (see, e.g., [22]). A great deal is known about such representations, a basic tool in their study being the use of the corresponding 'infinitesimal' representations of Lie algebras.

Formally, a representation of a Lie group G on a vector space V (over a field K)

[10]The dimensions given are dimensions over $\mathbb{C}$. Note that every complex Lie group/algebra can also be viewed as a real Lie group/algebra of twice the dimension.

is a group homomorphism $G \to Aut(V)$ from G to the automorphism group of V. If a basis for the vector space V is chosen, the representation can be expressed as a homomorphism into $GL(n, K)$. This is known as a *matrix representation.*

On the Lie algebra level, there is a corresponding linear map from the Lie algebra of G to $End(V)$ preserving the Lie bracket $[\cdot, \cdot]$.

If the homomorphism is in fact an monomorphism, the representation is said to be *faithful.*

A unitary representation is defined in the same way, except that G maps to unitary matrices; the Lie algebra will then map to skew–Hermitian matrices.

Now, if G is a semisimple group, its finite–dimensional representations can be decomposed as direct sums of irreducible representations. The irreducibles are indexed by highest weight; the allowable (*dominant*) highest weights satisfy a suitable positivity condition. In particular, there exists a set of *fundamental weights*, indexed by the vertices of the *Dynkin diagram* of G (see below), such that dominant weights are simply non–negative integer linear combinations of the fundamental weights.

If G is a commutative compact Lie group, then its irreducible representations are simply the continuous characters of G. A *quotient representation* is a quotient module of the group ring.

4.9 Root Systems and Dynkin Diagrams

A *root system* is a special configuration in Euclidean space that has turned out to be fundamental in Lie theory as well as in its applications. Also, the classification scheme for root systems, by *Dynkin diagrams*, occurs in parts of mathematics with no overt connection to Lie groups (such as singularity theory, see e.g., [22]).

4.9.1 Definitions

Formally, a *root system* is a finite set Φ of non–zero vectors (*roots*) spanning a finite–dimensional Euclidean space V and satisfying the following properties:

1. The only scalar multiples of a root α in V which belong to Φ are α itself and -α.

2. For every root α in V, the set Φ is symmetric under reflection through the hyperplane of vectors perpendicular to α.

3. If α and β are vectors in Φ, the projection of 2β onto the line through α is an integer multiple of α.

The *rank* of a root system Φ is the dimension of V. Two root systems may be combined by regarding the Euclidean spaces they span as mutually orthogonal

subspaces of a common Euclidean space. A root system which does not arise from such a combination, such as the systems A_2, B_2, and G_2 in Figure 3, is said to be *irreducible.*

Two irreducible root systems (V_1, Φ_1) and (V_2, Φ_2) are considered to be the same if there is an invertible linear transformation $V_1 \to V_2$ which preserves distance up to a scale factor and which sends Φ_1 to Φ_2.

The group of isometries of V generated by reflections through hyperplanes associated to the roots of Φ is called the Weyl group of Φ as it acts faithfully on the finite set Φ, the Weyl group is always finite.

4.9.2 Classification

It is not too difficult to classify the root systems of rank 2 (see Figure 3).

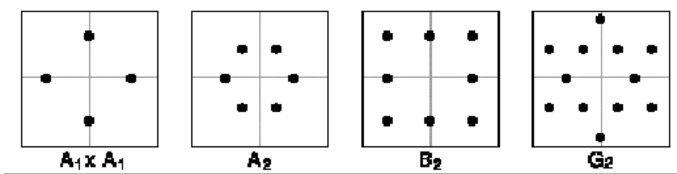

Figure 3: Classification of root systems of rank 2.

Whenever Φ is a root system in V and W is a subspace of V spanned by $\Psi = \Phi \cap W$, then Ψ is a root system in W. Thus, our exhaustive list of root systems of rank 2 shows the geometric possibilities for any two roots in a root system. In particular, two such roots meet at an angle of 0, 30, 45, 60, 90, 120, 135, 150, or 180 degrees.

In general, irreducible root systems are specified by a family (indicated by a letter A to G) and the rank (indicated by a subscript n). There are four *infinite families*:

- A_n $(n \geq 1)$, which corresponds to the special unitary group, $SU(n+1)$;
- B_n $(n \geq 2)$, which corresponds to the special orthogonal group, $SO(2n+1)$;
- C_n $(n \geq 3)$, which corresponds to the symplectic group, $Sp(2n)$;
- D_n $(n \geq 4)$, which corresponds to the special orthogonal group, $SO(2n)$,

as well as five *exceptional cases*: E_6, E_7, E_8, F_4, G_2.

4.9.3 Dynkin Diagrams

A Dynkin diagram is a graph with a few different kinds of possible edges (see Figure 4). The connected components of the graph correspond to the irreducible subalgebras of $\mathfrak{g}$. So a simple Lie algebra's Dynkin diagram has only one component. The rules are restrictive. In fact, there are only certain possibilities for each component, corresponding to the classification of semi–simple Lie algebras (see, e.g., [26]).

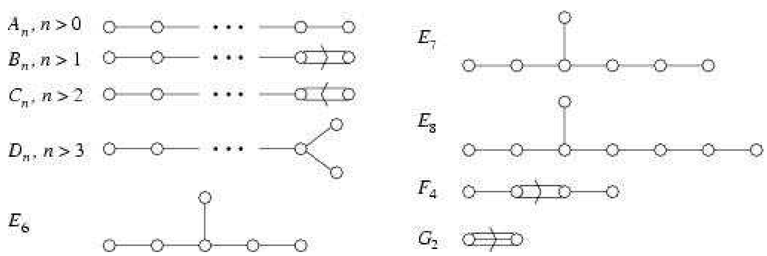

Figure 4: The problem of classifying irreducible root systems reduces to the problem of classifying connected Dynkin diagrams.

The *roots* of a complex Lie algebra form a lattice of rank k in a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, where k is the Lie algebra rank of $\mathfrak{g}$. Hence, the *root lattice* can be considered a lattice in $\mathbb{R}^k$. A vertex, or node, in the Dynkin diagram is drawn for each *Lie algebra simple root*, which corresponds to a generator of the root lattice. Between two nodes α and β, an edge is drawn if the simple roots are not perpendicular. One line is drawn if the angle between them is $2\pi/3$, two lines if the angle is $3\pi/4$, and three lines are drawn if the angle is $5\pi/6$. There are no other possible angles between Lie algebra simple roots. Alternatively, the number of lines N between the simple roots α and β is given by

$$N = A_{\alpha\beta} A_{\beta\alpha} = \frac{2\langle\alpha,\beta\rangle}{|\alpha|^2} \frac{2\langle\beta,\alpha\rangle}{|\beta|^2} = 4\cos^2\theta,$$

where $A_{\alpha\beta} = \frac{2\langle\alpha,\beta\rangle}{|\alpha|^2}$ is an entry in the *Cartan matrix* $(A_{\alpha\beta})$ (for details on Cartan matrix see, e.g., [22]). In a Dynkin diagram, an arrow is drawn from the longer root to the shorter root (when the angle is $3\pi/4$ or $5\pi/6$).

Here are some properties of *admissible Dynkin diagrams*:

1. A diagram obtained by removing a node from an admissible diagram is admissible.

2. An admissible diagram has no loops.

3. No node has more than three lines attached to it.

4. A sequence of nodes with only two single lines can be collapsed to give an admissible diagram.

5. The only connected diagram with a triple line has two nodes.

A *Coxeter–Dynkin diagram*, also called a *Coxeter graph*, is the same as a Dynkin diagram, but without the arrows. The Coxeter diagram is sufficient to characterize the algebra, as can be seen by enumerating connected diagrams.

The simplest way to recover a simple Lie algebra from its Dynkin diagram is to first reconstruct its Cartan matrix (A_{ij}). The ith node and jth node are connected by $A_{ij}A_{ji}$ lines. Since $A_{ij} = 0$ iff $A_{ji} = 0$, and otherwise $A_{ij} \in \{-3, -2, -1\}$, it is easy to find A_{ij} and A_{ji}, up to order, from their product. The arrow in the diagram indicates which is larger. For example, if node 1 and node 2 have two lines between them, from node 1 to node 2, then $A_{12} = -1$ and $A_{21} = -2$.

However, it is worth pointing out that each simple Lie algebra can be constructed concretely. For instance, the infinite families A_n, B_n, C_n, and D_n correspond to the special linear Lie algebra $\mathfrak{gl}(n+1, \mathbb{C})$, the odd orthogonal Lie algebra $\mathfrak{so}(2n+1, \mathbb{C})$, the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$, and the even orthogonal Lie algebra $\mathfrak{so}(2n, \mathbb{C})$. The other simple Lie algebras are called *exceptional Lie algebras*, and have constructions related to the *octonions*.

To prove this classification Theorem, one uses the angles between pairs of roots to encode the root system in a much simpler combinatorial object, the Dynkin diagram. The Dynkin diagrams can then be classified according to the scheme given above.

To every root system is associated a corresponding Dynkin diagram. Otherwise, the Dynkin diagram can be extracted from the root system by choosing a *base*, that is a subset Δ of Φ which is a basis of V with the special property that every vector in Φ when written in the basis Δ has either all coefficients ≥ 0 or else all ≤ 0.

The vertices of the Dynkin diagram correspond to vectors in Δ. An edge is drawn between each non–orthogonal pair of vectors; it is a double edge if they make an angle of 135 degrees, and a triple edge if they make an angle of 150 degrees. In addition, double and triple edges are marked with an angle sign pointing toward the shorter vector.

Although a given root system has more than one base, the Weyl group acts transitively on the set of bases. Therefore, the root system determines the Dynkin diagram. Given two root systems with the same Dynkin diagram, we can match up roots, starting with the roots in the base, and show that the systems are in fact the same.

Thus the problem of classifying root systems reduces to the problem of classifying possible Dynkin diagrams, and the problem of classifying irreducible root systems reduces to the problem of classifying connected Dynkin diagrams. Dynkin diagrams

encode the inner product on E in terms of the basis Δ, and the condition that this inner product must be positive definite turns out to be all that is needed to get the desired classification (see Figure 4).

In detail, the individual root systems can be realized case–by–case, as in the following paragraphs:

$\mathbf{A}_n$. Let V be the subspace of $\mathbb{R}^{n+1}$ for which the coordinates sum to 0, and let Φ be the set of vectors in V of length $\sqrt{2}$ and with integer coordinates in $\mathbb{R}^{n+1}$. Such a vector must have all but two coordinates equal to 0, one coordinate equal to 1, and one equal to -1, so there are $n^2 + n$ roots in all.

$\mathbf{B}_n$. Let $V = \mathbb{R}^n$, and let Φ consist of all integer vectors in V of length 1 or $\sqrt{2}$. The total number of roots is $2n^2$.

$\mathbf{C}_n$: Let $V = \mathbb{R}^n$, and let Φ consist of all integer vectors in V of $\sqrt{2}$ together with all vectors of the form 2λ, where λ is an integer vector of length 1. The total number of roots is $2n^2$. The total number of roots is $2n^2$.

$\mathbf{D}_n$. Let $V = \mathbb{R}^n$, and let Φ consist of all integer vectors in V of length $\sqrt{2}$. The total number of roots is $2n(n-1)$.

$\mathbf{E}_n$. For V_8, let $V = \mathbb{R}^8$, and let E_8 denote the set of vectors α of length $\sqrt{2}$ such that the coordinates of 2α are all integers and are either all even or all odd. Then E_7 can be constructed as the intersection of E_8 with the hyperplane of vectors perpendicular to a fixed root α in E_8, and E_6 can be constructed as the intersection of E_8 with two such hyperplanes corresponding to roots α and β which are neither orthogonal to one another nor scalar multiples of one another. The root systems E_6, E_7, and E_8 have 72, 126, and 240 roots respectively.

$\mathbf{F}_4$. For F_4, let $V = \mathbb{R}^4$, and let Φ denote the set of vectors α of length 1 or $\sqrt{2}$ such that the coordinates of 2α are all integers and are either all even or all odd. There are 48 roots in this system.

$\mathbf{G}_2$. There are 12 roots in G_2, which form the vertices of a *hexagram*.

4.9.4 Irreducible Root Systems

Irreducible root systems classify a number of related objects in Lie theory, notably:

1. Simple complex Lie algebras;
2. Simple complex Lie groups;
3. Simply connected complex Lie groups which are simple modulo centers; and
4. Simple compact Lie groups.

In each case, the roots are non–zero weights of the adjoint representation.

A root system can also be said to describe a *plant's root* and associated systems.

4.10 Simple and Semisimple Lie Groups and Algebras

A *simple Lie group* is a Lie group which is also a simple group. These groups, and groups closely related to them, include many of the so–called *classical groups* of geometry, which lie behind projective geometry and other geometries derived from it by the *Erlangen programme* of Felix Klein. They also include some *exceptional groups*, that were first discovered by those pursuing the classification of simple Lie groups. The exceptional groups account for many special examples and configurations in other branches of mathematics. In particular the classification of finite simple groups depended on a thorough prior knowledge of the 'exceptional' possibilities.

The complete listing of the simple Lie groups is the basis for the theory of the semisimple Lie groups and reductive groups, and their representation theory. This has turned out not only to be a major extension of the theory of compact Lie groups (and their representation theory), but to be of basic significance in mathematical physics.

Such groups are classified using the prior classification of the complex simple Lie algebras. It has been shown that a simple Lie group has a simple Lie algebra that will occur on the list given there, once it is complexified (that is, made into a complex vector space rather than a real one). This reduces the classification to two further matters.

The groups $SO(p, q, \mathbb{R})$ and $SO(p+q, \mathbb{R})$, for example, give rise to different real Lie algebras, but having the same Dynkin diagram. In general there may be different *real forms* of the same complex Lie algebra.

Secondly, the Lie algebra only determines uniquely the simply connected (universal) cover G^* of the component containing the identity of a Lie group G. It may well happen that G^* is not actually a simple group, for example having a non–trivial center. We have therefore to worry about the global topology, by computing the fundamental group of G (an Abelian group: a Lie group is an $H-$space). This was done by Elie Cartan.

For an example, take the special orthogonal groups in even dimension. With $-I$ a scalar matrix in the center, these are not actually simple groups; and having a two–fold spin cover, they aren't simply–connected either. They lie 'between' G^* and G, in the notation above.

Recall that a *semisimple module* is a module in which each submodule is a direct summand. In particular, a *semisimple representation* is completely reducible, i.e., is a direct sum of irreducible representations (under a descending chain condition). Similarly, one speaks of an Abelian category as being semisimple when every object has the corresponding property. Also, a semisimple ring is one that is semisimple as a module over itself.

A *semisimple matrix* is diagonalizable over any algebraically closed field containing its entries. In practice this means that it has a diagonal matrix as its Jordan normal form.

A *Lie algebra* $\mathfrak{g}$ is called *semisimple* when it is a direct sum of *simple Lie algebras*, i.e., non–trivial Lie algebras $\mathfrak{L}$ whose only ideals are $\{0\}$ and $\mathfrak{L}$ itself. An equivalent condition is that the *Killing form*

$$\mathcal{B}(X,Y) = \mathrm{Tr}(Ad(X)\,Ad(Y))$$

is non–degenerate [27]. The following properties can be proved equivalent for a finite–dimensional algebra $\mathfrak{L}$ over a field of characteristic 0:

1. $\mathfrak{L}$ is semisimple.
2. $\mathfrak{L}$ has no nonzero Abelian ideal.
3. $\mathfrak{L}$ has zero radical (the radical is the biggest solvable ideal).
4. Every representation of $\mathfrak{L}$ is fully reducible, i.e., is a sum of irreducible representations.
5. $\mathfrak{L}$ is a (finite) direct product of simple Lie algebras (a Lie algebra is called simple if it is not Abelian and has no nonzero ideal).

A *connected Lie group* is called *semisimple* when its Lie algebra is semisimple; and the same holds for algebraic groups. Every finite dimensional representation of a semisimple Lie algebra, Lie group, or algebraic group in characteristic 0 is semisimple, i.e., completely reducible, but the converse is not true. Moreover, in characteristic $p > 0$, semisimple Lie groups and Lie algebras have finite dimensional representations which are not semisimple. An element of a semisimple Lie group or Lie algebra is itself semisimple if its image in every finite–dimensional representation is semisimple in the sense of matrices.

Every semisimple Lie algebra $\mathfrak{g}$ can be classified by its Dynkin diagram [22].

5 Some Classical Examples of Lie Groups

5.1 Galilei Group

The *Galilei group* is the group of transformations in space and time that connect those Cartesian systems that are termed ‘inertial frames’ in Newtonian mechanics. The most general relationship between two such frames is the following. The origin of the time scale in the inertial frame S' may be shifted compared with that in S; the orientation of the Cartesian axes in S' may be different from that in S; the origin O of the Cartesian frame in S' may be moving relative to the origin O in S at a uniform velocity. The transition from S to S' involves ten parameters; thus the Galilei group is a ten parameter group. The basic assumption inherent in Galilei–Newtonian relativity is that there is an absolute time scale, so that the only way

in which the time variables used by two different 'inertial observers' could possibly differ is that the zero of time for one of them may be shifted relative to the zero of time for the other (see, e.g. [1, 8, 9]).

Galilei space–time structure involves the following three elements:

1. *World*, as a 4D affine space A^4. The points of A^4 are called *world points* or *events*. The parallel transitions of the world A^4 form a linear (i.e., Euclidean) space $\mathbb{R}^4$.

2. *Time*, as a linear map $t : \mathbb{R}^4 \to \mathbb{R}$ of the linear space of the world parallel transitions onto the real 'time axes'. Time interval from the event $a \in A^4$ to $b \in A^4$ is called the number $t(b-a)$; if $t(b-a) = 0$ then the events a and b are called synchronous. The set of all mutually synchronous events consists a 3D affine space A^3, being a subspace of the world A^4. The kernel of the mapping t consists of the parallel transitions of A^4 translating arbitrary (and every) event to the synchronous one; it is a linear 3D subspace $\mathbb{R}^3$ of the space $\mathbb{R}^4$.

3. *Distance* (*metric*) between the synchronous events,

$$\rho(a, b) = \parallel a - b \parallel, \qquad \text{for all} \quad a, b \in A^3,$$

given by the scalar product in $\mathbb{R}^3$. The distance transforms arbitrary space of synchronous events into the well known 3D Euclidean space E^3.

The space A^4, with the Galilei space–time structure on it, is called Galilei space. Galilei group is the group of all possible transformations of the Galilei space, preserving its structure. The elements of the Galilei group are called Galilei transformations. Therefore, Galilei transformations are affine transformations of the world A^4 preserving the time intervals and distances between the synchronous events.

The direct product $\mathbb{R} \times \mathbb{R}^3$, of the time axes with the 3D linear space R3 with a fixed Euclidean structure, has a natural Galilei structure. It is called Galilei coordinate system.

5.2 General Linear Group

The group of linear isomorphisms of $\mathbb{R}^n$ to $\mathbb{R}^n$ is a Lie group of dimension n^2, called the *general linear group* and denoted $Gl(n, \mathbb{R})$. It is a smooth manifold, since it is a subset of the vector space $L(\mathbb{R}^n, \mathbb{R}^n)$ of all linear maps of $\mathbb{R}^n$ to $\mathbb{R}^n$, as $Gl(n, \mathbb{R})$ is the inverse image of $\mathbb{R} \backslash \{0\}$ under the continuous map $A \mapsto \det A$ of $L(\mathbb{R}^n, \mathbb{R}^n)$ to $\mathbb{R}$. The group operation is composition (see, e.g. [1, 3, 8, 9]).

$$(A, B) \in Gl(n, \mathbb{R}) \times Gl(n, \mathbb{R}) \mapsto A \circ B \in Gl(n, \mathbb{R})$$

and the inverse map is

$$A \in Gl(n,\mathbb{R}) \mapsto A^{-1} \in Gl(n,\mathbb{R}).$$

If we choose a basis in $\mathbb{R}^n$, we can represent each element $A \in Gl(n,\mathbb{R})$ by an invertible $(n \times n)-$-matrix. The group operation is then matrix multiplication and the inversion is matrix inversion. The identity is the identity matrix I_n. The group operations are smooth since the formulas for the product and inverse of matrices are smooth in the matrix components.

The Lie algebra of $Gl(n,\mathbb{R})$ is $\mathfrak{gl}(n)$, the vector space $L(\mathbb{R}^n,\mathbb{R}^n)$ of all linear transformations of $\mathbb{R}^n$, with the commutator bracket

$$[A,B] = AB - BA.$$

For every $A \in L(\mathbb{R}^n,\mathbb{R}^n)$,

$$\gamma_A : t \in \mathbb{R} \mapsto \gamma_A(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i \in Gl(n,\mathbb{R})$$

is a one–parameter subgroup of $Gl(n,\mathbb{R})$, because

$$\gamma_A(0) = I \qquad \text{and} \qquad \dot{\gamma}_A(t) = \sum_{i=0}^{\infty} \frac{t^{i-1}}{(i-1)!} A^i = \gamma_A(t)\, A$$

Hence γ_A is an integral curve of the left–invariant vector–field X_A. Therefore, the exponential map is given by

$$\exp : A \in L(\mathbb{R}^n,\mathbb{R}^n) \mapsto \exp(A) \equiv \mathrm{e}^A = \gamma_A(1) = \sum_{i=0}^{\infty} \frac{A^i}{i!} \in Gl(n,\mathbb{R}).$$

For each $A \in Gl(n,\mathbb{R})$ the corresponding adjoint map

$$Ad_A : L(\mathbb{R}^n,\mathbb{R}^n) \to L(\mathbb{R}^n,\mathbb{R}^n)$$

is given by

$$Ad_A B = A \cdot B \cdot A^{-1}.$$

5.3 Rotational Lie Groups in Human/Humanoid Biomechanics

Local kinematics at each rotational robot or (synovial) human joint, is defined as a *group action* of an nD constrained rotational Lie group $SO(n)$ on the Euclidean space $\mathbb{R}^n$. In particular, there is an action of $SO(2)-$-group in uniaxial human joints

(cylindrical, or *hinge joints*, like knee and elbow) and an action of $SO(3)-$-group in three–axial human joints (spherical, or *ball–and–socket joints*, like hip, shoulder, neck, wrist and ankle). In both cases, $SO(n)$ acts, with its operators of rotation, on the vector $x = \{x^\mu\}$, $(i = 1, 2, 3)$ of external, Cartesian coordinates of the parent body–segment, depending, at the same time, on the vector $q = \{q^s\}$, $(s = 1, \cdots, n)$ on n group–parameters, i.e., joint angles (see [5, 6, 8, 9]).

Each joint rotation $R \in SO(n)$ defines a map

$$R : x^\mu \mapsto \dot{x}^\mu, \qquad R(x^\mu, q^s) = R_{q^s} x^\mu,$$

where $R_{q^s} \in SO(n)$ are joint group operators. The vector $v = \{v_s\}$, $(s = 1, \cdots, n)$ of n infinitesimal generators of these rotations, i.e., joint angular velocities, given by

$$v_s = -\left[\frac{\partial R(x^\mu, q^s)}{\partial q^s}\right]_{q=0} \frac{\partial}{\partial x^\mu}$$

constitute an nD Lie algebra $\mathfrak{so}(n)$ corresponding to the joint rotation group $SO(n)$. Conversely, each joint group operator R_{q^s}, representing a one–parameter subgroup of $SO(n)$, is defined as the exponential map of the corresponding joint group generator v_s

$$R_{q^s} = \exp(q^s v_s)$$

This exponential map represents a solution of the joint operator differential equation in the joint group–parameter space $\{q^s\}$

$$\frac{dR_{q^s}}{dq^s} = v_s R_{q^s}.$$

5.3.1 Uniaxial Group of Joint Rotations

The uniaxial joint rotation in a single Cartesian plane around a perpendicular axis, e.g., $xy-$plane about the z axis, by an internal joint angle θ, leads to the following transformation of the joint coordinates:

$$x' = x\cos\theta - y\sin\theta, \qquad y' = x\sin\theta + y\cos\theta.$$

In this way, the joint $SO(2)-$group, given by

$$SO(2) = \left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} | \theta \in [0, 2\pi] \right\},$$

acts in a canonical way on the Euclidean plane $\mathbb{R}^2$ by

$$SO(2) = \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \longmapsto \begin{pmatrix} x\cos\theta & -y\sin\theta \\ x\sin\theta & y\cos\theta \end{pmatrix}.$$

Its associated Lie algebra $\mathfrak{so}(2)$ is given by

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} | t \in \mathbb{R} \right\},$$

since the curve $\gamma_\theta \in SO(2)$ given by

$$\gamma_\theta : t \in \mathbb{R} \longmapsto \gamma_\theta(t) = \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix} \in SO(2),$$

passes through the identity $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and then

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_\theta(t) = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix},$$

so that I_2 is a basis of $\mathfrak{so}(2)$, since $\dim\left(SO(2)\right) = 1$.

The *exponential map* $\exp : \mathfrak{so}(2) \to SO(2)$ is given by

$$\exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \gamma_\theta(1) = \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix}.$$

The *infinitesimal generator* of the action of $SO(2)$ on $\mathbb{R}^2$, i.e., joint angular velocity v, is given by

$$v = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y},$$

since

$$v_{\mathbb{R}^2}\left(x, y\right) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv)\left(x, y\right) = \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} \cos tv & -\sin tv \\ \sin tv & \cos tv \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The *momentum map* $J : T^*\mathbb{R}^2 \to \mathbb{R}$ associated to the lifted action of $SO(2)$ on $T^*\mathbb{R}^2 \simeq \mathbb{R}^4$ is given by

$$\begin{aligned} J\left(x, y, p_1, p_2\right) &= xp_y - yp_x, \qquad \text{since} \\ J\left(x, y, p_x, p_y\right)(\xi) &= (p_x dx + p_y dy)(v_{\mathbb{R}^2}) = -vp_x y + -vp_y x. \end{aligned}$$

The Lie group $SO(2)$ acts on the symplectic manifold $(\mathbb{R}^4, \omega = dp_x \wedge dx + dp_y \wedge dx)$ by

$$\begin{aligned} &\phi\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, (x, y, p_x, p_y)\right) \\ = \;& \left(x\cos\theta - y\sin\theta,\, x\sin\theta + y\cos\theta,\, p_x\cos\theta - p_y\sin\theta,\, p_x\sin\theta + p_y\cos\theta\right). \end{aligned}$$

5.3.2 Three–Axial Group of Joint Rotations

The three–axial $SO(3)-$group of human–like joint rotations depends on three parameters, Euler joint angles $q^i = (\varphi, \psi, \theta)$, defining the rotations about the Cartesian coordinate triedar (x, y, z) placed at the joint pivot point. Each of the Euler angles are defined in the constrained range $(-\pi, \pi)$, so the joint group space is a constrained sphere of radius π (see [5, 6, 8, 9]).

Let $G = SO(3) = \{A \in \mathcal{M}_{3\times 3}(\mathbb{R}) : A^t A = I_3, \det(A) = 1\}$ be the group of rotations in $\mathbb{R}^3$. It is a Lie group and $\dim(G) = 3$. Let us isolate its one–parameter joint subgroups, i.e., consider the three operators of the finite joint rotations $R_\varphi, R_\psi, R_\theta \in SO(3)$, given by

$$R_\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, \quad R_\psi = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

corresponding respectively to rotations about $x-$axis by an angle φ, about $y-$axis by an angle ψ, and about $z-$axis by an angle θ.

The total three–axial joint rotation A is defined as the product of above one–parameter rotations $R_\varphi, R_\psi, R_\theta$, i.e., $A = R_\varphi \cdot R_\psi \cdot R_\theta$ is equal[11]

$$A = \begin{bmatrix} \cos\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & \cos\psi\cos\varphi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\sin\psi\cos\varphi - \cos\theta\sin\varphi\sin\psi & -\sin\psi\sin\varphi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix}.$$

However, the order of these matrix products matters: different order products give different results, as the matrix product is *noncommutative product*. This is the reason why Hamilton's *quaternions*[12] are today commonly used to parameterize the $SO(3)-$group, especially in the field of 3D computer graphics.

The one–parameter rotations $R_\varphi, R_\psi, R_\theta$ define curves in $SO(3)$ starting from $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Their derivatives in $\varphi = 0, \psi = 0$ and $\theta = 0$ belong to the associated *tangent Lie algebra* $\mathfrak{so}(3)$. That is the corresponding infinitesimal generators of joint rotations – joint angular velocities $v_\varphi, v_\psi, v_\theta \in \mathfrak{so}(3)$ – are respectively given

[11] Note that this product is noncommutative, so it really depends on the order of multiplications.

[12] Recall that the set of Hamilton's *quaternions* $\mathbb{H}$ represents an extension of the set of complex numbers $\mathbb{C}$. We can compute a rotation about the unit vector, $\mathbf{u}$ by an angle θ. The quaternion q that computes this rotation is

$$q = \left(\cos\frac{\theta}{2}\ ,\ u\sin\frac{\theta}{2}\right).$$

by

$$
\begin{aligned}
v_\varphi &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}, \qquad v_\psi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \\
v_\theta &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.
\end{aligned}
$$

Moreover, the elements are linearly independent and so

$$
\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -a & b \\ a & 0 & -\gamma \\ -b & \gamma & 0 \end{bmatrix} | a, b, \gamma \in \mathbb{R} \right\}.
$$

The Lie algebra $\mathfrak{so}(3)$ is identified with $\mathbb{R}^3$ by associating to each $v = (v_\varphi, v_\psi, v_\theta) \in \mathbb{R}^3$ the matrix $v \in \mathfrak{so}(3)$ given by $v = \begin{bmatrix} 0 & -a & b \\ a & 0 & -\gamma \\ -b & \gamma & 0 \end{bmatrix}$. Then we have the following identities:

1. $\widehat{u \times v} = [\hat{u}, v]$; and
2. $u \cdot v = -\frac{1}{2}\operatorname{Tr}(\hat{u} \cdot v)$.

The exponential map $\exp : \mathfrak{so}(3) \to SO(3)$ is given by *Rodrigues relation*

$$
\exp(v) = I + \frac{\sin\|v\|}{\|v\|}v + \frac{1}{2}\left(\frac{\sin\frac{\|v\|}{2}}{\frac{\|v\|}{2}}\right)^2 v^2
$$

where the norm $\|v\|$ is given by

$$
\|v\| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}.
$$

The the dual, *cotangent Lie algebra* $\mathfrak{so}(3)^*$, includes the three joint angular momenta $p_\varphi, p_\psi, p_\theta \in \mathfrak{so}(3)^*$, derived from the joint velocities v by multiplying them with corresponding moments of inertia.

Note that the parameterization of $SO(3)-$rotations is the subject of continuous research and development in many theoretical and applied fields of mechanics, such as rigid body, structural, and multibody dynamics, robotics, spacecraft attitude dynamics, navigation, image processing, etc.

5.3.3 The Heavy Top

Consider a rigid body moving with a fixed point but under the influence of gravity. This problem still has a configuration space $SO(3)$, but the symmetry group is only the circle group S^1, consisting of rotations about the direction of gravity. One says that gravity has broken the symmetry from $SO(3)$ to S^1. This time, eliminating the S^1 symmetry mysteriously leads one to the larger Euclidean group $SE(3)$ of rigid motion of $\mathbb{R}^3$. Conversely, we can start with $SE(3)$ as the configuration space for the rigid–body and 'reduce out' translations to arrive at $SO(3)$ as the configuration space. The equations of motion for a rigid body with a fixed point in a gravitational field give an interesting example of a system that is Hamiltonian. The underlying Lie algebra consists of the algebra of infinitesimal Euclidean motions in $\mathbb{R}^3$ (see [1, 3, 8, 9]).

The basic phase–space we start with is again $T^*SO(3)$, parameterized by Euler angles and their conjugate momenta. In these variables, the equations are in canonical Hamiltonian form. However, the presence of gravity breaks the symmetry, and the system is no longer $SO(3)$ invariant, so it cannot be written entirely in terms of the body angular momentum p. One also needs to keep track of Γ, the 'direction of gravity' as seen from the body. This is defined by $\Gamma = A^{-1}k$, where k points upward and A is the element of $SO(3)$ describing the current configuration of the body. The equations of motion are

$$\begin{aligned}
\dot{p}_1 &= \frac{I_2 - I_3}{I_2 I_3} p_2 p_3 + Mgl(\Gamma^2 \chi^3 - \Gamma^3 \chi^2), \\
\dot{p}_2 &= \frac{I_3 - I_1}{I_3 I_1} p_3 p_1 + Mgl(\Gamma^3 \chi^1 - \Gamma^1 \chi^3), \\
\dot{p}_3 &= \frac{I_1 - I_2}{I_1 I_2} p_1 p_2 + Mgl(\Gamma^1 \chi^2 - \Gamma^2 \chi^1), \\
&\text{and} \qquad \dot{\Gamma} \;=\; \Gamma \times \Omega,
\end{aligned}$$

where Ω is the body's angular velocity vector, I_1, I_2, I_3 are the body's principal moments of inertia, M is the body's mass, g is the acceleration of gravity, χ is the body fixed unit vector on the line segment connecting the fixed point with the body's center of mass, and l is the length of this segment.

5.4 Euclidean Groups of Rigid Body Motion

In this subsection we give description of two most important Lie groups in classical mechanics in 2D and 3D, $SE(2)$ and $SE(3)$, respectively (see [4, 8, 9]).

5.4.1 Special Euclidean Group $SE(2)$ in the Plane

The motion in uniaxial human joints is naturally modelled by the *special Euclidean group in the plane*, $SE(2)$. It consists of all transformations of $\mathbb{R}^2$ of the form $Az+a$, where $z, a \in \mathbb{R}^2$, and

$$A \in SO(2) = \left\{ \text{matrices of the form } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}.$$

In other words, group $SE(2)$ consists of matrices of the form:
$(R_\theta, a) = \begin{pmatrix} R_\theta & a \\ 0 & I \end{pmatrix}$, where $a \in \mathbb{R}^2$ and R_θ is the rotation matrix:
$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, while I is the 3×3 identity matrix. The inverse $(R_\theta, a)^{-1}$ is given by

$$(R_\theta, a)^{-1} = \begin{pmatrix} R_\theta & a \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} R_{-\theta} & -R_{-\theta}a \\ 0 & I \end{pmatrix}.$$

The Lie algebra $\mathfrak{se}(2)$ of $SE(2)$ consists of 3×3 block matrices of the form

$$\begin{pmatrix} -\xi J & v \\ 0 & 0 \end{pmatrix}, \qquad \text{where} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (J^T = J^{-1} = -J),$$

with the usual commutator bracket. If we identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ by the isomorphism

$$\begin{pmatrix} -\xi J & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(2) \longmapsto (\xi, v) \in \mathbb{R}^3,$$

then the expression for the Lie algebra bracket becomes

$$[(\xi, v_1, v_2), (\zeta, w_1, w_2)] = (0, \zeta v_2 - \xi w_2, \xi w_1 - \zeta v1) = (0, \xi J^T w - \zeta J^T v),$$

where $v = (v_1, v_2)$ and $w = (w_1, w_2)$.

The *adjoint group action* of

$$(R_\theta, a) \begin{pmatrix} R_\theta & a \\ 0 & I \end{pmatrix} \qquad \text{on} \qquad (\xi, v) = \begin{pmatrix} -\xi J & v \\ 0 & 0 \end{pmatrix}$$

is given by the *group conjugation*,

$$\begin{pmatrix} R_\theta & a \\ 0 & I \end{pmatrix} \begin{pmatrix} -\xi J & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{-\theta} & -R_{-\theta}a \\ 0 & I \end{pmatrix} = \begin{pmatrix} -\xi J & \xi J a + R_\theta v \\ 0 & 0 \end{pmatrix},$$

or, in coordinates,

$$Ad_{(R_\theta,a)}(\xi,v)=(\xi,\xi Ja+R_\theta v). \tag{2}$$

In proving (2) we used the identity $R_\theta J=JR_\theta$. Identify the dual algebra, $\mathfrak{se}(2)^*$, with matrices of the form $\begin{pmatrix} \frac{\mu}{2}J & 0 \\ \alpha & 0 \end{pmatrix}$, via the nondegenerate pairing given by the trace of the product. Thus, $\mathfrak{se}(2)^*$ is isomorphic to $\mathbb{R}^3$ via

$$\begin{pmatrix} \frac{\mu}{2}J & 0 \\ \alpha & 0 \end{pmatrix} \in \mathfrak{se}(2)^* \longmapsto (\mu,\alpha)\in\mathbb{R}^3,$$

so that in these coordinates, the pairing between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$ becomes

$$\langle(\mu,\alpha),(\xi,v)\rangle=\mu\xi+\alpha\cdot v,$$

that is, the usual dot product in $\mathbb{R}^3$. The *coadjoint group action* is thus given by

$$Ad^*_{(R_\theta,a)^{-1}}(\mu,\alpha)=(\mu-R_\theta\alpha\cdot Ja+R_\theta\alpha). \tag{3}$$

Formula (3) shows that the coadjoint orbits are the cylinders $T^*S^1_\alpha=\{(\mu,\alpha)|\,\|\alpha\|=\text{const}\}$ if $\alpha\neq0$ together with the points are on the $\mu-$axis. The canonical cotangent bundle projection $\pi:T^*S^1_\alpha\to S^1_\alpha$ is defined as $\pi(\mu,\alpha)=\alpha$.

5.4.2 Special Euclidean Group $SE(3)$ in the 3D Space

The most common group structure in human biodynamics is the *special Euclidean group in 3D space*, $SE(3)$. Briefly, the Euclidean $SE(3)-$group is defined as a semidirect (noncommutative) product of 3D rotations and 3D translations, $SE(3):=SO(3)\rhd\mathbb{R}^3$ (see [4, 8, 9]). Its most important subgroups are the following:

Subgroup	**Definition**
$SO(3)$, group of rotations in 3D (a spherical joint)	Set of all proper orthogonal $3\times3-$rotational matrices
$SE(2)$, special Euclidean group in 2D (all planar motions)	Set of all $3\times3-$matrices: $\begin{bmatrix}\cos\theta & \sin\theta & r_x\\ -\sin\theta & \cos\theta & r_y\\ 0 & 0 & 1\end{bmatrix}$
$SO(2)$, group of rotations in 2D subgroup of $SE(2)$–group (a revolute joint)	Set of all proper orthogonal $2\times2-$rotational matrices included in $SE(2)-$group
$\mathbb{R}^3$, group of translations in 3D (all spatial displacements)	Euclidean 3D vector space

Lie Group $SE(3)$ and Its Lie Algebra An element of $SE(3)$ is a pair (A, a) where $A \in SO(3)$ and $a \in \mathbb{R}^3$. The action of $SE(3)$ on $\mathbb{R}^3$ is the rotation A followed by translation by the vector a and has the expression

$$(A, a) \cdot x = Ax + a.$$

Using this formula, one sees that multiplication and inversion in $SE(3)$ are given by

$$(A, a)(B, b) = (AB, Ab + a) \qquad \text{and} \qquad (A, a)^{-1} = (A^{-1}, -A^{-1}a),$$

for $A, B \in SO(3)$ and $a, b \in \mathbb{R}^3$. The identity element is $(l, 0)$.

The Lie algebra of the Euclidean group $SE(3)$ is $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3$ with the Lie bracket

$$[(\xi, u), (\eta, v)] = (\xi \times \eta, \xi \times v - \eta \times u). \tag{4}$$

The Lie algebra of the Euclidean group has a structure that is a special case of what is called a *semidirect product*. Here it is the *product of the group of rotations with the corresponding group of translations*. It turns out that semidirect products occur under rather general circumstances when the symmetry in T^*G is broken (see [4, 8, 9]).

The dual Lie algebra of the Euclidean group $SE(3)$ is $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$ with the same Lie bracket (5).

Representation of $SE(3)$ In other words, $SE(3) := SO(3) \triangleright \mathbb{R}^3$ is the Lie group consisting of isometries of $\mathbb{R}^3$.

Using homogeneous coordinates, we can represent $SE(3)$ as follows,

$$SE(3) = \left\{ \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in GL(4, \mathbb{R}) : R \in SO(3),\, p \in \mathbb{R}^3 \right\},$$

with the action on $\mathbb{R}^3$ given by the usual matrix–vector product when we identify $\mathbb{R}^3$ with the section $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. In particular, given

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in SE(3),$$

and $q \in \mathbb{R}^3$, we have

$$g \cdot q = Rq + p,$$

or as a matrix–vector product,

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} Rq + p \\ 1 \end{pmatrix}.$$

Lie algebra of $SE(3)$ The Lie algebra of $SE(3)$ is given by

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \boldsymbol{\omega} & v \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}) : \boldsymbol{\omega} \in \mathfrak{so}(3),\, v \in \mathbb{R}^3 \right\},$$

where the attitude matrix $\boldsymbol{\omega} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$

The exponential map of $SE(3)$ The exponential map, $\exp : \mathfrak{se}(3) \to SE(3)$, is given by

$$\exp \begin{pmatrix} \boldsymbol{\omega} & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\boldsymbol{\omega}) & Av \\ 0 & 1 \end{pmatrix},$$

where

$$A = I + \frac{1 - \cos \|\omega\|}{\|\omega\|^2} \boldsymbol{\omega} + \frac{\|\omega\| - \sin \|\omega\|}{\|\omega\|^3} \boldsymbol{\omega}^2,$$

and $\exp(\boldsymbol{\omega})$ is given by the Rodriguez' formula,

$$\exp(\boldsymbol{\omega}) = I + \frac{\sin \|\omega\|}{\|\omega\|} \boldsymbol{\omega} + \frac{1 - \cos \|\omega\|}{\|\omega\|^2} \boldsymbol{\omega}^2.$$

In other words, the special Euclidean group $SE(3) := SO(3) \rhd \mathbb{R}^3$ is the Lie group consisting of isometries of the Euclidean 3D space $\mathbb{R}^3$. An element of $SE(3)$ is a pair (A, a) where $A \in SO(3)$ and $a \in \mathbb{R}^3$. The action of $SE(3)$ on $\mathbb{R}^3$ is the rotation A followed by translation by the vector a and has the expression

$$(A, a) \cdot x = Ax + a.$$

The Lie algebra of the Euclidean group $SE(3)$ is $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3$ with the Lie bracket

$$[(\xi, u), (\eta, v)] = (\xi \times \eta, \xi \times v - \eta \times u). \tag{5}$$

Using homogeneous coordinates, we can represent $SE(3)$ as follows,

$$SE(3) = \left\{ \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in GL(4, \mathbb{R}) : R \in SO(3),\, p \in \mathbb{R}^3 \right\},$$

with the action on $\mathbb{R}^3$ given by the usual matrix–vector product when we identify $\mathbb{R}^3$ with the section $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. In particular, given

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in SE(3),$$

and $q \in \mathbb{R}^3$, we have

$$g \cdot q = Rq + p,$$

or as a matrix–vector product,

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} Rq + p \\ 1 \end{pmatrix}.$$

The Lie algebra of $SE(3)$, denoted $\mathfrak{se}(3)$, is given by

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \omega & v \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}) : \omega \in \mathfrak{so}(3),\, v \in \mathbb{R}^3 \right\},$$

where the attitude (or, angular velocity) matrix $\omega : \mathbb{R}^3 \to \mathfrak{so}(3)$ is given by

$$\omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$

The *exponential map*, $\exp : \mathfrak{se}(3) \to SE(3)$, is given by

$$\exp \begin{pmatrix} \omega & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\omega) & Av \\ 0 & 1 \end{pmatrix},$$

where

$$A = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\omega + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3}\omega^2,$$

and $\exp(\omega)$ is given by the *Rodriguez' formula*,

$$\exp(\omega) = I + \frac{\sin\|\omega\|}{\|\omega\|}\omega + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\omega^2.$$

5.5 Basic Mechanical Examples

5.5.1 $SE(2)-$Hovercraft

Configuration manifold is $(\theta, x, y) \in SE(2)$, given by matrix

$$P = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Kinematic equations of motion in Lie algebra $\mathfrak{se}(2)$:

$$\dot{P} = P \begin{bmatrix} 0 & \omega & v_x \\ -\omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}, \qquad (\omega = \dot{\theta},\, v_x = \dot{x}, v_y = \dot{y}).$$

Kinetic energy:

$$E_k = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}I\omega^2,$$

where m, I are mass and inertia moment of the hovercraft.

Dynamical equations of motion:

$$\begin{aligned} m\dot{v}_x &= m\omega v_y + u_1, \\ m\dot{v}_y &= -m\omega v_x + u_2, \\ I\dot{\omega} &= \tau u_2, \end{aligned}$$

where $\tau = -h$ is the torque applied at distance h from the center–of–mass, while u_1, u_2 are control inputs.

5.5.2 $SO(3)$−Satellite

Configuration manifold is rotation matrix $R \in SO(3)$, with associated angular–velocity (attitude) matrix $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \in \mathfrak{so}(3) \approx \mathbb{R}^3$ given by

$$\boldsymbol{\omega} \in \mathfrak{so}(3) \longmapsto \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Kinematic equation of motion in $\mathfrak{so}(3)$:

$$\dot{R} = R\boldsymbol{\omega},$$

Kinetic energy:

$$E_k = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega},$$

where inertia tensor $\mathbf{I}$ is given by diagonal matrix,

$$\mathbf{I} = diag\{I_1, I_2, I_3\}.$$

Dynamical Euler equations of motion:

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \tau_i u^i,$$

where $\times$ is the cross–product in 3D, τ_i are three external torques and $u^i = u^i(t)$ are control inputs.

5.5.3 $SE(3)-$Submarine

The motion of a rigid body in incompressible, irrotational and inviscid fluid is defined by the configuration manifold $SE(3)$, given by a pair of rotation matrix and translation vector, $(R, p) \in SE(3)$, such that angular velocity (attitude) matrix and linear velocity vector, $(\boldsymbol{\omega}, \mathbf{v}) \in \mathfrak{se}(3) \approx \mathbb{R}^6$.

Kinematic equations of motion in $\mathfrak{se}(3)$:

$$\dot{p} = R\mathbf{v}, \qquad \dot{R} = R\boldsymbol{\omega}.$$

Kinetic energy (symmetrical):

$$E_k = \frac{1}{2}\mathbf{v}^T\mathbf{M}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\omega},$$

where mass and inertia matrices are diagonal (for a neutrally buoyant ellipsoidal body with uniformly distributed mass),

$$\begin{aligned} \mathbf{M} &= diag\{m_1, m_2, m_3\}, \\ \mathbf{I} &= diag\{I_1, I_2, I_3\}. \end{aligned}$$

Dynamical *Kirchhoff equations of motion* read:

$$\mathbf{M}\dot{\mathbf{v}} = \mathbf{M}\mathbf{v} \times \boldsymbol{\omega}, \qquad \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{M}\mathbf{v} \times \mathbf{v}.$$

By including the body–fixed external forces and torques, f_i, τ_i, with input controls $u^i = u^(t)$, the dynamical equations become:

$$\begin{aligned} \mathbf{M}\dot{\mathbf{v}} &= \mathbf{M}\mathbf{v} \times \boldsymbol{\omega} + f_i u^i, \\ \mathbf{I}\dot{\boldsymbol{\omega}} &= \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{M}\mathbf{v} \times \mathbf{v} + \tau_i u^i. \end{aligned}$$

5.6 Newton–Euler $SE(3)-$Dynamics

5.6.1 $SO(3)$: Euler Equations of Rigid Rotations

Unforced Euler equations read in vector form

$$\dot{\boldsymbol{\pi}} \equiv \mathbf{I}\dot{\boldsymbol{\omega}} = \boldsymbol{\pi} \times \boldsymbol{\omega}, \qquad \text{with} \quad \mathbf{I} = diag\{I_1, I_2, I_3\}$$

and in scalar form

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1. \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

Using rotational kinetic–energy Lagrangian

$$L(\boldsymbol{\omega}) = E_k^{rot} = \frac{1}{2}\boldsymbol{\omega}^t \mathbf{I}\boldsymbol{\omega} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \qquad (^t\text{= 'transpose'})$$

Regarding the angular momentum $\boldsymbol{\pi} = \partial_{\boldsymbol{\omega}} L = \mathbf{I}\boldsymbol{\omega} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ as a vector, we can derive unforced Euler equations: $\dot{\boldsymbol{\pi}} = \boldsymbol{\pi} \times \boldsymbol{\omega}$ as a system of *Euler–Lagrange–Kirchhoff equations*

$$\frac{d}{dt}\partial_{\boldsymbol{\omega}} L = \partial_{\boldsymbol{\omega}} L \times \boldsymbol{\omega}.$$

Forced Euler equations read in vector form

$$\dot{\boldsymbol{\pi}} + \boldsymbol{\omega} \times \boldsymbol{\pi} = \mathbf{T}$$

and in scalar form

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= T_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= T_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= T_3 \end{aligned}$$

5.6.2 $SE(3)$: Coupled Newton–Euler Equations

Forced coupled Newton–Euler equations read in vector form

$$\begin{aligned} &\dot{\mathbf{p}} \equiv \mathbf{M}\dot{\mathbf{v}} = \mathbf{F} + \mathbf{p} \times \boldsymbol{\omega}, \qquad \text{with } \mathbf{M} = diag\{m_1, m_2, m_3\} \\ &\dot{\boldsymbol{\pi}} \equiv \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{T} + \boldsymbol{\pi} \times \boldsymbol{\omega} + \mathbf{p} \times \mathbf{v}, \qquad \mathbf{I} = diag\{I_1, I_2, I_3\}, \end{aligned}$$

with principal inertia moments given in Cartesian coordinates (x, y, z) by density ρ−dependent volume integrals

$$I_1 = \iiint \rho(z^2+y^2)dxdydz, \;\; I_2 = \iiint \rho(x^2+y^2)dxdydz, \;\; I_3 = \iiint \rho(x^2+y^2)dxdydz,$$

In tensor form, the forced–coupled Newton–Euler equations read

$$\begin{aligned} \dot{p}_i &\equiv M_{ij}\dot{v}^j = F_i + \varepsilon^j_{ik} p_j \omega^k, \\ \dot{\pi}_i &\equiv I_{ij}\dot{\omega}^j = T_i + \varepsilon^j_{ik} \pi_j \omega^k + \varepsilon^j_{ik} p_j v^k, \end{aligned}$$

where the permutation symbol ε^j_{ik} is defined as

$$\varepsilon^j_{ik} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } k = i. \end{cases}$$

In scalar form these equations read

$$\begin{aligned}
\dot{p}_1 &= F_1 - m_3 v_3 \omega_2 + m_2 v_2 \omega_3 \\
\dot{p}_2 &= F_2 + m_3 v_3 \omega_1 - m_1 v_1 \omega_3 \\
\dot{p}_3 &= F_3 - m_2 v_2 \omega_1 + m_1 v_1 \omega_2
\end{aligned}$$

$$\begin{aligned}
\dot{\pi}_1 &= T_1 + (m_2 - m_3) v_2 v_3 + (I_2 - I_3)\omega_2 \omega_3 \\
\dot{\pi}_2 &= T_2 + (m_3 - m_1) v_1 v_3 + (I_3 - I_1)\omega_1 \omega_3 \\
\dot{\pi}_3 &= T_3 + (m_1 - m_2) v_1 v_2 + (I_1 - I_2)\omega_1 \omega_2
\end{aligned}.$$

These coupled rigid–body equations can be derived from the *Newton–Euler kinetic energy*

$$E_k = \frac{1}{2}\mathbf{v}^t \mathbf{M}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}^t \mathbf{I}\boldsymbol{\omega}$$

or, in tensor form

$$E = \frac{1}{2} M_{ij} \dot{v}^i \dot{v}^j + \frac{1}{2} I_{ij} \dot{\omega}^i \dot{\omega}^j.$$

Using the *Kirchhoff–Lagrangian equations*

$$\begin{aligned}
\frac{d}{dt}\partial_{\mathbf{v}} E_k &= \partial_{\mathbf{v}} E_k \times \boldsymbol{\omega} + \mathbf{F} \\
\frac{d}{dt}\partial_{\boldsymbol{\omega}} E_k &= \partial_{\boldsymbol{\omega}} E_k \times \boldsymbol{\omega} + \partial_{\mathbf{v}} E_k \times \mathbf{v} + \mathbf{T},
\end{aligned}$$

or, in tensor form

$$\begin{aligned}
\frac{d}{dt}\partial_{v^i} E &= \varepsilon^j_{ik} \left(\partial_{v^j} E\right) \omega^k + F_i, \\
\frac{d}{dt}\partial_{\omega^i} E &= \varepsilon^j_{ik} \left(\partial_{\omega^j} E\right) \omega^k + \varepsilon^j_{ik} \left(\partial_{v^j} E\right) v^k + T_i
\end{aligned}$$

we can derive linear and angular momentum covectors

$$\mathbf{p} = \partial_{\mathbf{v}} E_k, \qquad \boldsymbol{\pi} = \partial_{\boldsymbol{\omega}} E_k$$

or, in tensor form

$$p_i = \partial_{v^i} E, \qquad \pi_i = \partial_{\omega^i} E,$$

and in scalar form

$$\begin{aligned}
\mathbf{p} &= [p_1, p_2, p_3] = [m_1 v_1, m_2 v_2, m_2 v_3] \\
\boldsymbol{\pi} &= [\pi_1, \pi_2, \pi_3] = [I_1 \omega_1, I_2 \omega_2, I_3 \omega_3],
\end{aligned}$$

with their respective time derivatives, in vector form

$$\dot{\mathbf{p}} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}\partial_{\mathbf{v}} E_k, \qquad \dot{\boldsymbol{\pi}} = \frac{d}{dt}\boldsymbol{\pi} = \frac{d}{dt}\partial_{\boldsymbol{\omega}} E_k$$

or, in tensor form

$$\dot{p}_i = \frac{d}{dt}p_i = \frac{d}{dt}\partial_{v^i} E, \qquad \dot{\pi}_i = \frac{d}{dt}\pi_i = \frac{d}{dt}\partial_{\omega^i} E,$$

and in scalar form

$$\begin{aligned} \dot{\mathbf{p}} &= [\dot{p}_1, \dot{p}_2, \dot{p}_3] = [m_1\dot{v}_1, m_2\dot{v}_2, m_3\dot{v}_3] \\ \dot{\boldsymbol{\pi}} &= [\dot{\pi}_1, \dot{\pi}_2, \dot{\pi}_3] = [I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3]. \end{aligned}$$

In addition, for the purpose of biomechanical injury prediction/prevention, we have linear and angular jolts, respectively given in vector form by

$$\begin{aligned} &\dot{\mathbf{F}} = \ddot{\mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\omega} - \mathbf{p} \times \dot{\boldsymbol{\omega}} \\ &\dot{\mathbf{T}} = \ddot{\boldsymbol{\pi}} - \dot{\boldsymbol{\pi}} \times \boldsymbol{\omega} - \boldsymbol{\pi} \times \dot{\boldsymbol{\omega}} - \dot{\mathbf{p}} \times \mathbf{v} - \mathbf{p} \times \dot{\mathbf{v}}, \end{aligned}$$

or, in tensor form[13]

$$\begin{aligned} \dot{F}_i &= \ddot{p}_i - \varepsilon^j_{ik}\dot{p}_j\omega^k - \varepsilon^j_{ik}p_j\dot{\omega}^k, \\ \dot{T}_i &= \ddot{\pi}_i - \varepsilon^j_{ik}\dot{\pi}_j\omega^k - \varepsilon^j_{ik}\pi_j\dot{\omega}^k - \varepsilon^j_{ik}\dot{p}_j v^k - \varepsilon^j_{ik}p_j\dot{v}^k, \end{aligned}$$

where the linear and angular jolt covectors are

$$\begin{aligned} \dot{\mathbf{F}} &\equiv \dot{F}_i = \mathbf{M}\ddot{\mathbf{v}} \equiv M_{ij}\ddot{v}^j = [\dot{F}_1, \dot{F}_2, \dot{F}_3], \\ \dot{\mathbf{T}} &\equiv \dot{T}_i = \mathbf{I}\ddot{\boldsymbol{\omega}} \equiv I_{ij}\ddot{\omega}^j = [\dot{T}_1, \dot{T}_2, \dot{T}_3], \end{aligned}$$

where

$$\ddot{\mathbf{v}} = \ddot{v}^i = [\ddot{v}^1, \ddot{v}^2, \ddot{v}^3]^t, \qquad \ddot{\boldsymbol{\omega}} = \ddot{\omega}^i = [\ddot{\omega}^1, \ddot{\omega}^2, \ddot{\omega}^3]^t,$$

are linear and angular jerk vectors.

In scalar form, the $SE(3)-$jolt expands as

$$\left\{ \begin{array}{l} \dot{F}_1 = \ddot{p}_1 - m_2\omega_3\dot{v}_2 + m_3\left(\omega_2\dot{v}_3 + v_3\dot{\omega}_2\right) - m_2 v_2\dot{\omega}_3, \\ \dot{F}_2 = \ddot{p}_2 + m_1\omega_3\dot{v}_1 - m_3\omega_1\dot{v}_3 - m_3 v_3\dot{\omega}_1 + m_1 v_1\dot{\omega}_3, \\ \dot{F}_3 = \ddot{p}_3 - m_1\omega_2\dot{v}_1 + m_2\omega_1\dot{v}_2 - v_2\dot{\omega}_1 - m_1 v_1\dot{\omega}_2, \end{array} \right.$$

$$\left\{ \begin{array}{l} \dot{T}_1 = \ddot{\pi}_1 - (m_2 - m_3)\left(v_3\dot{v}_2 + v_2\dot{v}_3\right) - (I_2 - I_3)\left(\omega_3\dot{\omega}_2 + \omega_2\dot{\omega}_3\right), \\ \dot{T}_2 = \ddot{\pi}_2 + (m_1 - m_3)\left(v_3\dot{v}_1 + v_1\dot{v}_3\right) + (I_1 - I_3)\left(\omega_3\dot{\omega}_1 + \omega_1\dot{\omega}_3\right), \\ \dot{T}_3 = \ddot{\pi}_3 - (m_1 - m_2)\left(v_2\dot{v}_1 + v_1\dot{v}_2\right) - (I_1 - I_2)\left(\omega_2\dot{\omega}_1 + \omega_1\dot{\omega}_2\right). \end{array} \right.$$

[13]In this paragraph the overdots actually denote the absolute Bianchi (covariant) derivatives, so that the jolts retain the proper covector character, which would be lost if ordinary time derivatives are used. However, for simplicity, we stick to the same notation.

5.7 Symplectic Group in Hamiltonian Mechanics

Here we give a brief description of symplectic group (see [4, 8, 9]).

Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, with I the $n \times n$ identity matrix. Now, $A \in L(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is called a *symplectic matrix* if $A^T J\, A = J$. Let $Sp(2n, \mathbb{R})$ be the set of $2n \times 2n$ symplectic matrices. Taking determinants of the condition $A^T J\, A = J$ gives $\det A = \pm 1$, and so $A \in GL(2n, \mathbb{R})$. Furthermore, if $A, B \in Sp(2n, R)$, then $(AB)^T J(AB) = B^T A^T JAB = J$. Hence, $AB \in Sp(2n, \mathbb{R})$, and if $A^T J\, A = J$, then $JA = (A^T)^{-1} J = (A^{-1})^T J$, so $J = (A-1)^T\, JA^{-1}$, or $A^{-1} \in Sp(2n, \mathbb{R})$. Thus, $Sp(2n, \mathbb{R})$ is a group.

The *symplectic Lie group*

$$Sp(2n, \mathbb{R}) = \left\{ A \in GL(2n, \mathbb{R}) : A^T J\, A = J \right\}$$

is a noncompact, connected Lie group of dimension $2n^2 + n$. Its Lie algebra

$$\mathfrak{sp}(2n, \mathbb{R}) = \left\{ A \in L(\mathbb{R}^{2n}, \mathbb{R}^{2n}) : A^T J\, A = J = 0 \right\},$$

called the *symplectic Lie algebra*, consists of the $2n \times 2n$ matrices A satisfying $A^T J\, A = 0$.

Consider a particle of mass m moving in a potential $V(q)$, where $q^i = (q^1, q^2, q^3) \in \mathbb{R}^3$. Newtonian second law states that the particle moves along a curve $q(t)$ in $\mathbb{R}^3$ in such a way that $m\ddot{q}^i = -\operatorname{grad} V(q^i)$. Introduce the 3D–momentum $p_i = m\dot{q}^i$, and the energy (Hamiltonian)

$$H(q, p) = \frac{1}{2m} \sum_{i=1}^{3} p_i^2 + V(q).$$

Then

$$\begin{aligned} \frac{\partial H}{\partial q^i} &= \frac{\partial V}{\partial q^i} = -m\ddot{q}^i = -\dot{p}_i, \quad \text{and} \\ \frac{\partial H}{\partial p_i} &= \frac{1}{m} p_i = \dot{q}^i, \qquad (i = 1, 2, 3), \end{aligned}$$

and hence Newtonian law $\boxed{F = m\ddot{q}^i}$ is equivalent to *Hamiltonian equations*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Now, writing $z = (q^i, p_i)$,

$$J \operatorname{grad} H(z) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \left(\dot{q}^i, \dot{p}_i \right) = \dot{z},$$

so the *complex Hamiltonian equations* read

$$\dot{z} = J \operatorname{grad} H(z).$$

Now let $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ and write $w = f(z)$. If $z(t)$ satisfies the complex Hamiltonian equations then $w(t) = f(z(t))$ satisfies $\dot{w} = A^T \dot{z}$, where $A^T = [\partial w^i / \partial z^j]$ is the Jacobian matrix of f. By the chain rule,

$$\dot{w} = A^T J \operatorname*{grad}_z H(z) = A^T J \, A \operatorname*{grad}_w H(z(w)).$$

Thus, the equations for $w(t)$ have the form of Hamiltonian equations with energy $K(w) = H(z(w))$ iff $A^T J \, A = J$, that is, iff A is symplectic. A nonlinear transformation f is canonical iff its Jacobian matrix is symplectic. $Sp(2n, \mathbb{R})$ is the linear invariance group of classical mechanics.

6 Medical Applications: Prediction of Injuries

6.1 General Theory of Musculo–Skeletal Injury Mechanics

The prediction and prevention of traumatic brain injury, spinal injury and musculo-skeletal injury is a very important aspect of preventive medical science. Recently, in a series of papers [28, 29, 30], we have proposed a new coupled loading-rate hypothesis as a unique cause of all above injuries. This new hypothesis states that the unique cause of brain, spinal and musculo-skeletal injuries is a Euclidean Jolt, which is an impulsive loading that strikes any part of the human body (head, spine or any bone/joint) – in several coupled degrees-of-freedom simultaneously. It never goes in a single direction only. Also, it is never a static force. It is always an impulsive translational and/or rotational force coupled to some mass eccentricity. This is, in a nutshell, our universal Jolt theory of all mechanical injuries.

To show this, based on the previously defined covariant force law, we have firstly formulated the fully coupled Newton–Euler dynamics of:

1. Brain's micro-motions within the cerebrospinal fluid inside the cranial cavity;
2. Any local inter-vertebral motions along the spine; and
3. Any local joint motions in the human musculo-skeletal system.

Then, from it, we have defined the essential concept of **Euclidean Jolt**, which is the main cause of all mechanical injuries. The Euclidean Jolt has two main components:

1. Sudden motion, caused either by an accidental impact or slightly distorted human movement; and

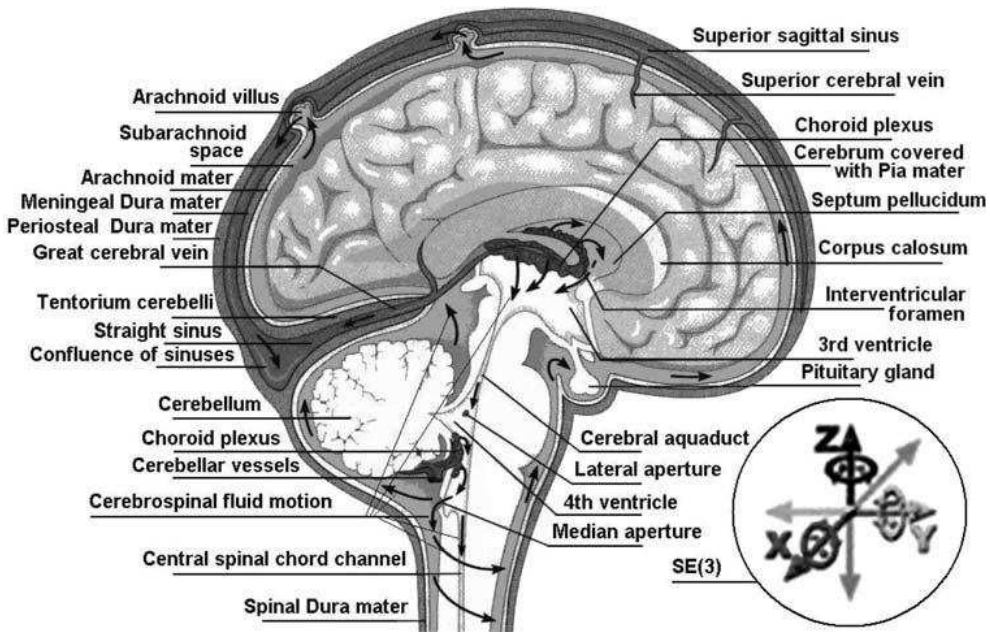

Figure 5: Human brain and its $SE(3)-$group of microscopic three-dimensional motions within the cerebrospinal fluid inside the cranial cavity.

2. Unnatural mass distribution of the human body (possibly with some added masses), which causes some mass eccentricity from the natural physiological body state.

What does this all mean? We will try to explain it in “plain English”. As we live in a 3D space, one could think that motion of any part of the human body, either caused by an accidental impact or by voluntary human movement, “just obeys classical mechanics in 6 degrees-of-freedom: three translations and three rotations”. However, these 6 degrees-of-freedom are not independent motions as it is suggested by the standard term “degrees-of-freedom”. In reality, these six motions of any body in space are coupled. Firstly, three rotations are coupled in the so-called rotation group (or matrix, or quaternion). Secondly, three translations are coupled with the rotation group to give the full Euclidean group of rigid body motions in space. A simple way to see this is to observe someone throwing an object in the air or hitting a tennis ball: how far and where it will fly depends not only on the standard “projectile” mechanics, but also on its local “spin” around all three axes simultaneously. Every golf and tennis player knows this simple fact. Once the spin is properly defined we have a “fully coupled Newton–Euler dynamics” – to start with.

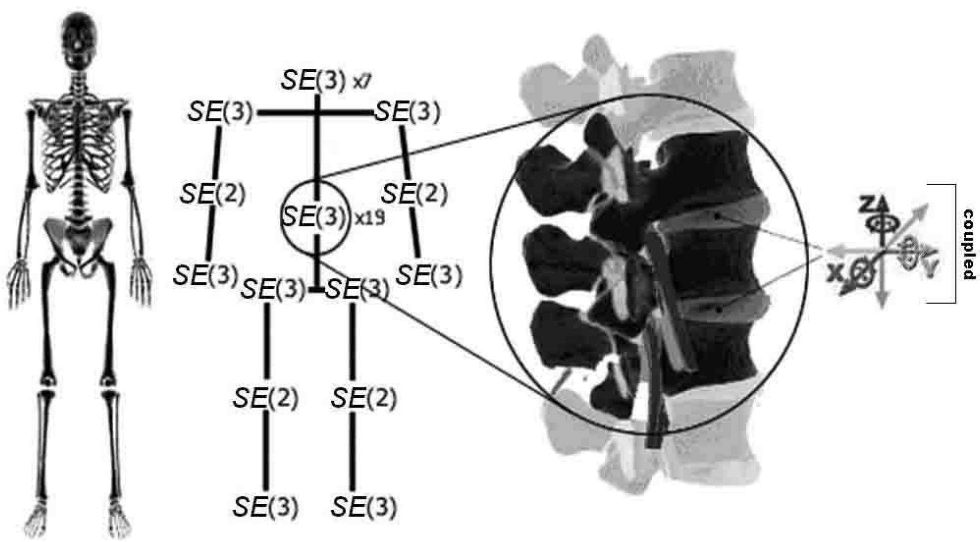

Figure 6: Human body representation in terms of SE(3)/SE(2)-groups of rigid-body motion, with the vertebral column represented as a chain of 26 flexibly-coupled SE(3)-groups.

The covariant force law for any biodynamical system (which we introduced earlier in our biodynamics books and papers, see our references in the cited papers above) goes one step beyond the Newton–Euler dynamics. It states:

Euclidean Force covector field $=$
Body mass distribution $\times$ **Euclidean Acceleration vector field**

This is a nontrivial biomechanical generalization of the fundamental Newton's definition of the force acting on a single particle. Unlike classical engineering mechanics of multi-body systems, this fundamental law of biomechanics proposes that forces acting on a multi-body system and causing its motions are fundamentally different physical quantities from the resulting accelerations. In simple words, forces are massive quantities while accelerations are massless quantities. More precisely, the acceleration vector field includes all linear and angular accelerations of individual body segments. When we couple them all with the total body's mass-distribution matrix of all body segments (including all masses and inertia moments), we get the force co-vector field, comprising all the forces and torques acting on the individual body segments. In this way, we have defined the 6-dimensional Euclidean force for an arbitrary biomechanical system.

Now, for prediction of injuries, we need to take the rate-of-change (or derivative,

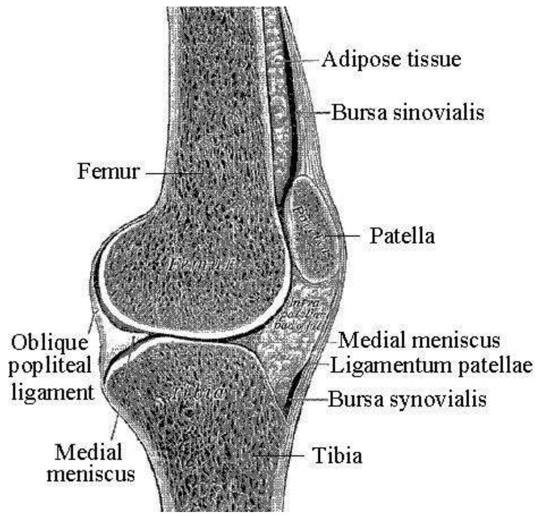

Figure 7: Schematic latero-frontal view of the left knee joint. Although designed to perform mainly flexion/extension (strictly in the sagittal plane) with some restricted medial/lateral rotation in the semi-flexed position, it is clear that the knee joint really has at least six-degrees-of-freedom, including three micro-translations. The injury actually occurs when some of these microscopic translations become macroscopic, which normally happens only after an external jolt.

with respect to time) of the Euclidean biomechanical force defined above. In this way, we get the Euclidean Jolt, which is the sudden change (in time) of the 6-dimensional Euclidean force:

$$\textbf{Euclidean Jolt covector field} = \textbf{Body mass distribution} \times \textbf{Euclidean Jerk vector field}$$

And again, it consists of two components: (i) massless linear and angular jerks (of all included body segments), and (ii) their mass distribution. For the sake of simplicity, we can say that the mass distribution matrix includes all involved segmental masses and inertia moments, as well as "eccentricities" or "pathological leverages" from the normal physiological state.

Therefore, the unique cause of all brain, spine and musculo-skeletal injuries has two components:

1. Coupled linear and angular jerks; and
2. Mass distribution with "eccentricities".

In other words, **there are no injuries in static conditions without any mass eccentricities; all injuries are caused by mutually coupled linear and angular jerks, which are also coupled with the involved human mass distribution.**

The Euclidean Jolt causes two forms of discontinuous brain, spine or musculo-skeletal injury:

1. Mild rotational disclinations; and
2. Severe translational dislocations (or, fractures).

In the cited papers above, we have developed the soft-body dynamics of biomechanical disclinations and dislocations, caused by the Euclidean Jolt, using the Cosserat multipolar viscoelastic continuum model.

Implications of the new universal theory are various, as follows.

A. The research in traumatic brain injury (TBI, see Figure 5) has so far identified the rotation of the brain-stem as the main cause of the TBI due to various crashes/impacts. The contribution of our universal Jolt theory to the TBI research is the following:

1. Rigorously defined this brain rotation as a mechanical disclination of the brain-stem tissue modelled by the Cosserat multipolar soft-body model;
2. Showing that brain rotation is never uni-axial but always three-axial;
3. Showing that brain rotation is always coupled with translational dislocations. This is a straightforward consequence of our universal Jolt theory.

These apparently 'obvious' facts are actually *radically new:* we cannot separately analyze rapid brain's rotations from translations, because they are in reality always coupled.

One practical application of the brain Jolt theory is in design of helmets. Briefly, a 'hard' helmet saves the skull but not the brain; alternatively, a 'soft' helmet protects the brain from the collision jolt but does not protect the skull. A good helmet is both 'hard' and 'soft'. A proper helmet would need to have both a hard external shell (to protect the skull) and a soft internal part (that will dissipate the energy from the collision jolt by its own destruction, in the same way as a car saves its passengers from the collision jolt by its own destruction).

Similarly, in designing safer car air-bags, the two critical points will be (i) their

placement within the car, and (ii) their "soft-hard characteristics", similar to the helmet characteristics described above.

B. In case of spinal injury (see Figure 6), the contribution of our universal Jolt theory is the following:

1. The spinal injury is always localized at the certain vertebral or inter-vertebral point;
2. In case of severe translational injuries (vertebral fractures or discus herniae) they can be identified using X-ray or other medical imaging scans; in case of microscopic rotational injuries (causing the back-pain syndrome) they cannot be identified using current medical imaging scans;
3. There is no spinal injury without one of the following two causes:
   a. Impulsive rotational + translational loading caused by either fast human movements or various crashes/impacts; and/or
   b. Static eccentricity from the normal physiological spinal form, caused by external loading;
   c. Any spinal injury is caused by a combination of the two points above: impulsive rotational + translational loading and static eccentricity.

This is a straightforward consequence of our universal Jolt theory. We cannot separately analyze translational and rotational spinal injuries. Also, there are no "static injuries" without eccentricity. Indian women have for centuries carried bulky loads on their heads without any spinal injuries; they just prevented any load eccentricities and any jerks in their motion.

The currently used "Principal loading hypothesis" that describes spinal injuries in terms of spinal tension, compression, bending, and shear, covers only a small subset of all spinal injuries covered by our universal Jolt theory. To prevent spinal injuries we need to develop spinal jolt awareness: ability to control all possible impulsive spinal loadings as well as static eccentricities.

C. In case of general musculo-skeletal injury (see Figure 7 for the particular case of knee injury), the contribution of our universal Jolt theory is the following:

1. The injury is always localized at the certain joint or bone and caused by an impulsive loading, which hits this particular joint/bone in several coupled degrees-of-freedom simultaneously;
2. Injury happens when most of the body mass is hanging on that joint; for example, in case of a knee injury, when most of the body mass is on one leg with

a semi-flexed knee — and then, caused by some external shock, the knee suddenly "jerks" (this can happen in running, skiing, and ball games, as well as various crashes/impacts); or, in case of shoulder injury, when most of the body mass is hanging on one arm and then it suddenly jerks.

To prevent these injuries we need to develop musculo-skeletal jolt awareness. For example, never overload a flexed knee and avoid any kind of uncontrolled motions (like slipping) or collisions with external objects.

6.2 Analytical Mechanics of Traumatic Brain Injury (TBI)

6.2.1 The $SE(3)$–jolt: the cause of TBI

In this subsection we give a brief on TBI mechanics. For more details and references, see [28].

In the language of modern dynamics, the microscopic motion of human brain within the skull is governed by the Euclidean SE(3)–group of 3D motions (see next subsection). Within brain's SE(3)–group we have both SE(3)–kinematics (consisting of SE(3)–velocity and its two time derivatives: SE(3)–acceleration and SE(3)–jerk) and SE(3)–dynamics (consisting of SE(3)–momentum and its two time derivatives: SE(3)–force and SE(3)–jolt), which is brain's kinematics × brain's mass–inertia distribution.

Informally, the external SE(3)–jolt[14] is a sharp and sudden change in the SE(3)–force acting on brain's mass–inertia distribution (given by brain's mass and inertia matrices). That is, a 'delta'–change in a 3D force–vector coupled to a 3D torque–vector, striking the head–shell with the brain immersed into the cerebrospinal fluid. In other words, the SE(3)–jolt is a sudden, sharp and discontinues shock in all 6 coupled dimensions of brain's continuous micro–motion within the cerebrospinal fluid (Figure 5), namely within the three Cartesian (x, y, z)–translations and the three corresponding Euler angles around the Cartesian axes: roll, pitch and yaw. If the SE(3)–jolt produces a mild shock to the brain (e.g., strong head shake), it causes mild TBI, with temporary disabled associated sensory-motor and/or cognitive functions and affecting respiration and movement. If the SE(3)–jolt produces a hard

[14]The mechanical SE(3)–jolt concept is based on the mathematical concept of higher–order tangency (rigorously defined in terms of jet bundles of the head's configuration manifold), as follows: When something hits the human head, or the head hits some external body, we have a collision. This is naturally described by the SE(3)–momentum, which is a nonlinear coupling of 3 linear Newtonian momenta with 3 angular Eulerian momenta. The tangent to the SE(3)–momentum, defined by the (absolute) time derivative, is the SE(3)–force. The second-order tangency is given by the SE(3)–jolt, which is the tangent to the SE(3)–force, also defined by the time derivative.

shock (hitting the head with external mass), it causes severe TBI, with the total loss of gesture, speech and movement.

The SE(3)–jolt is the absolute time–derivative of the covariant force 1–form (or, co-vector field). The fundamental law of biomechanics is the *covariant force law*:

Force co-vector field = Mass distribution × Acceleration vector–field,

which is formally written (using the Einstein summation convention, with indices labelling the three Cartesian translations and the three corresponding Euler angles):

$$F_\mu = m_{\mu\nu} a^\nu, \qquad (\mu, \nu = 1, ..., 6)$$

where F_μ denotes the 6 covariant components of the external "pushing" SE(3)–force co-vector field, $m_{\mu\nu}$ represents the 6×6 covariant components of brain's inertia–metric tensor, while a^ν corresponds to the 6 contravariant components of brain's internal SE(3)–acceleration vector-field.

Now, the covariant (absolute, Bianchi) time-derivative $\frac{D}{dt}(\cdot)$ of the covariant SE(3)–force F_μ defines the corresponding external "striking" SE(3)–jolt co-vector field:

$$\frac{D}{dt}(F_\mu) = m_{\mu\nu} \frac{D}{dt}(a^\nu) = m_{\mu\nu} \left(\dot{a}^\nu + \Gamma^\nu_{\mu\lambda} a^\mu a^\lambda \right), \tag{6}$$

where $\frac{D}{dt}(a^\nu)$ denotes the 6 contravariant components of brain's internal SE(3)–jerk vector-field and overdot $(\dot{\ })$ denotes the time derivative. $\Gamma^\nu_{\mu\lambda}$ are the Christoffel's symbols of the Levi–Civita connection for the SE(3)–group, which are zero in case of pure Cartesian translations and nonzero in case of rotations as well as in the full–coupling of translations and rotations.

In the following, we elaborate on the SE(3)–jolt concept (using vector and tensor methods) and its biophysical TBI consequences in the form of brain's dislocations and disclinations.

6.2.2 $SE(3)$–group of brain's micro–motions within the CSF

The brain and the CSF together exhibit periodic microscopic translational and rotational motion in a pulsatile fashion to and from the cranial cavity, in the frequency range of normal heart rate (with associated periodic squeezing of brain's ventricles). This micro–motion is mathematically defined by the Euclidean (gauge) $SE(3)$–group.

In other words, the gauge $SE(3)$–group of Euclidean micro-motions of the brain immersed in the cerebrospinal fluid within the cranial cavity, contains matrices of the form $\begin{pmatrix} \mathbf{R} & \mathbf{b} \\ 0 & 1 \end{pmatrix}$, where $\mathbf{b}$ is brain's 3D micro-translation vector and $\mathbf{R}$ is brain's 3D rotation matrix, given by the product $\mathbf{R} = R_\varphi \cdot R_\psi \cdot R_\theta$ of brain's three Eulerian

micro-rotations, roll $= R_\varphi$, pitch $= R_\psi$, yaw $= R_\theta$, performed respectively about the $x-$axis by an angle φ, about the $y-$axis by an angle ψ, and about the $z-$axis by an angle θ,

$$R_\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}, \; R_\psi = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix}, \; R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, brain's natural $SE(3)-$dynamics within the cerebrospinal fluid is given by the coupling of Newtonian (translational) and Eulerian (rotational) equations of micro-motion.

6.2.3 Brain's natural $SE(3)-$dynamics

To support our coupled loading–rate hypothesis, we formulate the coupled Newton–Euler dynamics of brain's micro-motions within the scull's $SE(3)-$group of motions. The forced Newton–Euler equations read in vector (boldface) form

$$\begin{aligned} \text{Newton} &: \quad \dot{\mathbf{p}} \equiv \mathbf{M}\dot{\mathbf{v}} = \mathbf{F} + \mathbf{p} \times \boldsymbol{\omega}, \\ \text{Euler} &: \quad \dot{\boldsymbol{\pi}} \equiv \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{T} + \boldsymbol{\pi} \times \boldsymbol{\omega} + \mathbf{p} \times \mathbf{v}, \end{aligned} \tag{7}$$

where $\times$ denotes the vector cross product,[15]

$$\mathbf{M} \equiv M_{ij} = diag\{m_1, m_2, m_3\} \qquad \text{and} \qquad \mathbf{I} \equiv I_{ij} = diag\{I_1, I_2, I_3\}, \qquad (i, j = 1, 2, 3)$$

are brain's (diagonal) mass and inertia matrices,[16] defining brain's mass–inertia distribution, with principal inertia moments given in Cartesian coordinates (x, y, z) by volume integrals

$$I_1 = \iiint \rho(z^2+y^2)dxdydz, \; I_2 = \iiint \rho(x^2+z^2)dxdydz, \; I_3 = \iiint \rho(x^2+y^2)dxdydz,$$

[15]Recall that the cross product $\mathbf{u} \times \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$ equals $\mathbf{u} \times \mathbf{v} = uv\sin\theta\mathbf{n}$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, while $\mathbf{n}$ is a unit vector perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$ and $\mathbf{v}$ form a right-handed system.

[16]In reality, mass and inertia matrices $(\mathbf{M}, \mathbf{I})$ are not diagonal but rather full 3×3 positive–definite symmetric matrices with coupled mass– and inertia–products. Even more realistic, fully–coupled mass–inertial properties of a brain immersed in (incompressible, irrotational and inviscid) cerebrospinal fluid are defined by the single non-diagonal 6×6 positive–definite symmetric mass–inertia matrix $\mathcal{M}_{SE(3)}$, the so-called material metric tensor of the $SE(3)-$group, which has all nonzero mass–inertia coupling products. In other words, the 6×6 matrix $\mathcal{M}_{SE(3)}$ contains: (i) brain's own mass plus the added mass matrix associated with the fluid, (ii) brain's own inertia plus the added inertia matrix associated with the potential flow of the fluid, and (iii) all the coupling terms between linear and angular momenta. However, for simplicity, in this paper we shall consider only the simple case of two separate diagonal 3×3 matrices $(\mathbf{M}, \mathbf{I})$.

dependent on brain's density $\rho = \rho(x, y, z)$,

$$\mathbf{v} \equiv v^i = [v_1, v_2, v_3]^t \qquad \text{and} \qquad \boldsymbol{\omega} \equiv \omega^i = [\omega_1, \omega_2, \omega_3]^t$$

(where $[\ ]^t$ denotes the vector transpose) are brain's linear and angular velocity vectors[17] (that is, column vectors),

$$\mathbf{F} \equiv F_i = [F_1, F_2, F_3] \qquad \text{and} \qquad \mathbf{T} \equiv T_i = [T_1, T_2, T_3]$$

are gravitational and other external force and torque co-vectors (that is, row vectors) acting on the brain within the scull,

$$\begin{array}{rcl} \mathbf{p} & \equiv & p_i \equiv \mathbf{M}\mathbf{v} = [p_1, p_2, p_3] = [m_1 v_1, m_2 v_2, m_2 v_2] \qquad \text{and} \\ \boldsymbol{\pi} & \equiv & \pi_i \equiv \mathbf{I}\boldsymbol{\omega} = [\pi_1, \pi_2, \pi_3] = [I_1 \omega_1, I_2 \omega_2, I_3 \omega_3] \end{array}$$

are brain's linear and angular momentum co-vectors.

In tensor form, the forced Newton–Euler equations (7) read

$$\begin{array}{rcl} \dot{p}_i & \equiv & M_{ij} \dot{v}^j = F_i + \varepsilon^j_{ik} p_j \omega^k, \qquad (i, j, k = 1, 2, 3) \\ \dot{\pi}_i & \equiv & I_{ij} \dot{\omega}^j = T_i + \varepsilon^j_{ik} \pi_j \omega^k + \varepsilon^j_{ik} p_j v^k, \end{array}$$

where the permutation symbol ε^j_{ik} is defined as

$$\varepsilon^j_{ik} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (3, 1, 2) \text{ or } (2, 3, 1), \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (1, 3, 2) \text{ or } (2, 1, 3), \\ 0 & \text{otherwise: } i = j \text{ or } j = k \text{ or } k = i. \end{cases}$$

In scalar form, the forced Newton–Euler equations (7) expand as

$$\begin{array}{rl} \text{Newton} : & \begin{cases} \dot{p}_1 = F_1 - m_3 v_3 \omega_2 + m_2 v_2 \omega_3 \\ \dot{p}_2 = F_2 + m_3 v_3 \omega_1 - m_1 v_1 \omega_3 \\ \dot{p}_3 = F_3 - m_2 v_2 \omega_1 + m_1 v_1 \omega_2 \end{cases}, \\ \text{Euler} : & \begin{cases} \dot{\pi}_1 = T_1 + (m_2 - m_3) v_2 v_3 + (I_2 - I_3) \omega_2 \omega_3 \\ \dot{\pi}_2 = T_2 + (m_3 - m_1) v_1 v_3 + (I_3 - I_1) \omega_1 \omega_3 \\ \dot{\pi}_3 = T_3 + (m_1 - m_2) v_1 v_2 + (I_1 - I_2) \omega_1 \omega_2 \end{cases}, \end{array} \tag{8}$$

showing brain's individual mass and inertia couplings.

[17]In reality, $\boldsymbol{\omega}$ is a 3×3 *attitude matrix*. However, for simplicity, we will stick to the (mostly) symmetrical translation–rotation vector form.

Equations (7)–(8) can be derived from the translational + rotational kinetic energy of the brain[18]

$$E_k = \frac{1}{2}\mathbf{v}^t\mathbf{M}\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}^t\mathbf{I}\boldsymbol{\omega}, \tag{9}$$

or, in tensor form

$$E = \frac{1}{2}M_{ij}v^iv^j + \frac{1}{2}I_{ij}\omega^i\omega^j.$$

For this we use the *Kirchhoff–Lagrangian equations*:

$$\begin{aligned} \frac{d}{dt}\partial_{\mathbf{v}}E_k &= \partial_{\mathbf{v}}E_k \times \boldsymbol{\omega} + \mathbf{F}, \\ \frac{d}{dt}\partial_{\boldsymbol{\omega}}E_k &= \partial_{\boldsymbol{\omega}}E_k \times \boldsymbol{\omega} + \partial_{\mathbf{v}}E_k \times \mathbf{v} + \mathbf{T}, \end{aligned} \tag{10}$$

where $\partial_{\mathbf{v}}E_k = \frac{\partial E_k}{\partial \mathbf{v}}$, $\partial_{\boldsymbol{\omega}}E_k = \frac{\partial E_k}{\partial \boldsymbol{\omega}}$; in tensor form these equations read

$$\begin{aligned} \frac{d}{dt}\partial_{v^i}E &= \varepsilon^j_{ik}\left(\partial_{v^j}E\right)\omega^k + F_i, \\ \frac{d}{dt}\partial_{\omega^i}E &= \varepsilon^j_{ik}\left(\partial_{\omega^j}E\right)\omega^k + \varepsilon^j_{ik}\left(\partial_{v^j}E\right)v^k + T_i. \end{aligned}$$

Using (9)–(10), brain's linear and angular momentum co-vectors are defined as

$$\mathbf{p} = \partial_{\mathbf{v}}E_k, \qquad \boldsymbol{\pi} = \partial_{\boldsymbol{\omega}}E_k,$$

or, in tensor form

$$p_i = \partial_{v^i}E, \qquad \pi_i = \partial_{\omega^i}E,$$

with their corresponding time derivatives, in vector form

$$\dot{\mathbf{p}} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}\partial_{\mathbf{v}}E, \qquad \dot{\boldsymbol{\pi}} = \frac{d}{dt}\boldsymbol{\pi} = \frac{d}{dt}\partial_{\boldsymbol{\omega}}E,$$

or, in tensor form

$$\dot{p}_i = \frac{d}{dt}p_i = \frac{d}{dt}\partial_{v^i}E, \qquad \dot{\pi}_i = \frac{d}{dt}\pi_i = \frac{d}{dt}\partial_{\omega^i}E,$$

[18] In a fully–coupled Newton–Euler brain dynamics, instead of equation (9) we would have brain's kinetic energy defined by the inner product:

$$E_k = \frac{1}{2}\left[\begin{pmatrix}\mathbf{p}\\ \boldsymbol{\pi}\end{pmatrix} |\mathcal{M}_{SE(3)} \begin{pmatrix}\mathbf{p}\\ \boldsymbol{\pi}\end{pmatrix}\right].$$

or, in scalar form

$$\dot{\mathbf{p}} = [\dot{p}_1, \dot{p}_2, \dot{p}_3] = [m_1\dot{v}_1, m_2\dot{v}_2, m_3\dot{v}_3], \qquad \dot{\boldsymbol{\pi}} = [\dot{\pi}_1, \dot{\pi}_2, \dot{\pi}_3] = [I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3].$$

While brain's healthy $SE(3)-$dynamics within the cerebrospinal fluid is given by the coupled Newton–Euler micro–dynamics, the TBI is actually caused by the sharp and discontinuous change in this natural $SE(3)$ micro-dynamics, in the form of the $SE(3)-$jolt, causing brain's discontinuous deformations.

6.2.4 Brain's traumatic dynamics: the $SE(3)-$jolt

The $SE(3)-$jolt, the actual cause of the TBI (in the form of the brain's plastic deformations), is defined as a coupled Newton+Euler jolt; in (co)vector form the $SE(3)-$jolt reads[19]

$$SE(3)-\text{jolt}: \begin{cases} \text{Newton jolt}: \dot{\mathbf{F}} = \ddot{\mathbf{p}} - \dot{\mathbf{p}} \times \boldsymbol{\omega} - \mathbf{p} \times \dot{\boldsymbol{\omega}} \ , \\ \text{Euler jolt}: \dot{\mathbf{T}} = \ddot{\boldsymbol{\pi}} \ - \dot{\boldsymbol{\pi}} \times \boldsymbol{\omega} - \boldsymbol{\pi} \times \dot{\boldsymbol{\omega}} - \dot{\mathbf{p}} \times \mathbf{v} - \mathbf{p} \times \dot{\mathbf{v}}, \end{cases}$$

where the linear and angular jolt co-vectors are

$$\dot{\mathbf{F}} \equiv \mathbf{M}\ddot{\mathbf{v}} = [\dot{F}_1, \dot{F}_2, \dot{F}_3], \qquad \dot{\mathbf{T}} \equiv \mathbf{I}\ddot{\boldsymbol{\omega}} = [\dot{T}_1, \dot{T}_2, \dot{T}_3],$$

where

$$\ddot{\mathbf{v}} = [\ddot{v}_1, \ddot{v}_2, \ddot{v}_3]^t, \qquad \ddot{\boldsymbol{\omega}} = [\ddot{\omega}_1, \ddot{\omega}_2, \ddot{\omega}_3]^t,$$

are linear and angular jerk vectors.

In tensor form, the $SE(3)-$jolt reads[20]

$$\begin{aligned} \dot{F}_i &= \ddot{p}_i - \varepsilon^j_{ik}\dot{p}_j\omega^k - \varepsilon^j_{ik}p_j\dot{\omega}^k, \qquad (i,j,k = 1,2,3) \\ \dot{T}_i &= \ddot{\pi}_i \ - \varepsilon^j_{ik}\dot{\pi}_j\omega^k - \varepsilon^j_{ik}\pi_j\dot{\omega}^k - \varepsilon^j_{ik}\dot{p}_j v^k - \varepsilon^j_{ik}p_j\dot{v}^k, \end{aligned}$$

in which the linear and angular jolt covectors are defined as

$$\begin{aligned} \dot{\mathbf{F}} &\equiv \dot{F}_i = \mathbf{M}\ddot{\mathbf{v}} \equiv M_{ij}\ddot{v}^j = [\dot{F}_1, \dot{F}_2, \dot{F}_3], \\ \dot{\mathbf{T}} &\equiv \dot{T}_i = \mathbf{I}\ddot{\boldsymbol{\omega}} \equiv I_{ij}\ddot{\omega}^j = [\dot{T}_1, \dot{T}_2, \dot{T}_3], \end{aligned}$$

where $\ddot{\mathbf{v}} = \ddot{v}^i$, and $\ddot{\boldsymbol{\omega}} = \ddot{\omega}^i$ are linear and angular jerk vectors.

[19] Note that the derivative of the cross–product of two vectors follows the standard calculus product–rule: $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \dot{\mathbf{u}} \times \mathbf{v} + \mathbf{u} \times \dot{\mathbf{v}}$.

[20] In this paragraph the overdots actually denote the absolute Bianchi (covariant) time-derivative (6), so that the jolts retain the proper covector character, which would be lost if ordinary time derivatives are used. However, for the sake of simplicity and wider readability, we stick to the same overdot notation.

In scalar form, the $SE(3)-$jolt expands as

$$\text{Newton jolt} \quad : \quad \begin{cases} \dot{F}_1 = \ddot{p}_1 - m_2\omega_3\dot{v}_2 + m_3\left(\omega_2\dot{v}_3 + v_3\dot{\omega}_2\right) - m_2 v_2\dot{\omega}_3, \\ \dot{F}_2 = \ddot{p}_2 + m_1\omega_3\dot{v}_1 - m_3\omega_1\dot{v}_3 - m_3 v_3\dot{\omega}_1 + m_1 v_1\dot{\omega}_3, \\ \dot{F}_3 = \ddot{p}_3 - m_1\omega_2\dot{v}_1 + m_2\omega_1\dot{v}_2 - v_2\dot{\omega}_1 - m_1 v_1\dot{\omega}_2, \end{cases}$$

$$\text{Euler jolt} \quad : \quad \begin{cases} \dot{T}_1 = \ddot{\pi}_1 - (m_2 - m_3)\left(v_3\dot{v}_2 + v_2\dot{v}_3\right) - (I_2 - I_3)\left(\omega_3\dot{\omega}_2 + \omega_2\dot{\omega}_3\right), \\ \dot{T}_2 = \ddot{\pi}_2 + (m_1 - m_3)\left(v_3\dot{v}_1 + v_1\dot{v}_3\right) + (I_1 - I_3)\left(\omega_3\dot{\omega}_1 + \omega_1\dot{\omega}_3\right), \\ \dot{T}_3 = \ddot{\pi}_3 - (m_1 - m_2)\left(v_2\dot{v}_1 + v_1\dot{v}_2\right) - (I_1 - I_2)\left(\omega_2\dot{\omega}_1 + \omega_1\dot{\omega}_2\right). \end{cases}$$

We remark here that the linear and angular momenta $(\mathbf{p}, \boldsymbol{\pi})$, forces $(\mathbf{F}, \mathbf{T})$ and jolts $(\dot{\mathbf{F}}, \dot{\mathbf{T}})$ are co-vectors (row vectors), while the linear and angular velocities $(\mathbf{v}, \boldsymbol{\omega})$, accelerations $(\dot{\mathbf{v}}, \dot{\boldsymbol{\omega}})$ and jerks $(\ddot{\mathbf{v}}, \ddot{\boldsymbol{\omega}})$ are vectors (column vectors). This bio-physically means that the 'jerk' vector should not be confused with the 'jolt' co-vector. For example, the 'jerk' means shaking the head's own mass–inertia matrices (mainly in the atlanto–occipital and atlanto–axial joints), while the 'jolt'means actually hitting the head with some external mass–inertia matrices included in the 'hitting' SE(3)–jolt, or hitting some external static/massive body with the head (e.g., the ground – gravitational effect, or the wall – inertial effect). Consequently, the mass-less 'jerk' vector represents a (translational+rotational) *non-collision effect* that can cause only weaker brain injuries, while the inertial 'jolt' co-vector represents a (translational+rotational) *collision effect* that can cause hard brain injuries.

6.2.5 Brain's dislocations and disclinations caused by the $SE(3)-$jolt

Recall from introduction that for mild TBI, the best injury predictor is considered to be the product of brain's strain and strain rate, which is the standard isotropic viscoelastic continuum concept. To improve this standard concept, in this subsection, we consider human brain as a 3D anisotropic multipolar *Cosserat viscoelastic continuum*, exhibiting coupled–stress–strain elastic properties. This non-standard continuum model is suitable for analyzing plastic (irreversible) deformations and fracture mechanics in multi-layered materials with microstructure (in which slips and bending of layers introduces additional degrees of freedom, non-existent in the standard continuum models.

The $SE(3)-$jolt $(\dot{\mathbf{F}}, \dot{\mathbf{T}})$ causes two types of brain's rapid discontinuous deformations:

1. The Newton jolt $\dot{\mathbf{F}}$ can cause micro-translational *dislocations*, or discontinuities in the Cosserat translations;

2. The Euler jolt $\dot{\mathbf{T}}$ can cause micro-rotational *disclinations*, or discontinuities in the Cosserat rotations.

To precisely define brain's dislocations and disclinations, caused by the $SE(3)-$jolt $(\dot{\mathbf{F}}, \dot{\mathbf{T}})$, we first define the coordinate co-frame, i.e., the set of basis 1–forms $\{dx^i\}$, given in local coordinates $x^i = (x^1, x^2, x^3) = (x, y, z)$, attached to brain's center-of-mass. Then, in the coordinate co-frame $\{dx^i\}$ we introduce the following set of brain's plastic–deformation–related $SE(3)-$based differential $p-$forms[21]:

the *dislocation current* 1–form, $\mathbf{J} = J_i\, dx^i$;
the *dislocation density* 2–form, $\boldsymbol{\alpha} = \frac{1}{2}\alpha_{ij}\, dx^i \wedge dx^j$;
the *disclination current* 2–form, $\mathbf{S} = \frac{1}{2}S_{ij}\, dx^i \wedge dx^j$; and
the *disclination density* 3–form, $\mathbf{Q} = \frac{1}{3!}Q_{ijk}\, dx^i \wedge dx^j \wedge dx^k$,

where $\wedge$ denotes the exterior wedge–product. These four $SE(3)-$based differential forms satisfy the following set of continuity equations:

$$\dot{\boldsymbol{\alpha}} = -\mathbf{dJ} - \mathbf{S}, \tag{11}$$

$$\dot{\mathbf{Q}} = -\mathbf{dS}, \tag{12}$$

$$\mathbf{d}\boldsymbol{\alpha} = \mathbf{Q}, \tag{13}$$

$$\mathbf{dQ} = \mathbf{0}, \tag{14}$$

where $\mathbf{d}$ denotes the exterior derivative.

In components, the simplest, fourth equation (14), representing the *Bianchi iden-*

[21] Differential $p-$forms are totally skew-symmetric covariant tensors, defined using the exterior wedge–product and exterior derivative. The proper definition of exterior derivative d for a $p-$form β on a smooth manifold M, includes the *Poincaré lemma*: $d(d\beta) = 0$, and validates the *general Stokes formula*

$$\int_{\partial M} \beta = \int_M d\beta,$$

where M is a $p-$dimensional *manifold with a boundary* and ∂M is its $(p-1)-$dimensional *boundary*, while the integrals have appropriate dimensions.

A $p-$form β is called *closed* if its exterior derivative is equal to zero,

$$d\beta = 0.$$

From this condition one can see that the closed form (the *kernel* of the exterior derivative operator d) is conserved quantity. Therefore, closed $p-$forms possess certain invariant properties, physically corresponding to the conservation laws.

A $p-$form β that is an exterior derivative of some $(p-1)-$form α,

$$\beta = d\alpha,$$

is called *exact* (the *image* of the exterior derivative operator d). By *Poincaré lemma*, exact forms prove to be closed automatically,

$$d\beta = d(d\alpha) = 0.$$

This lemma is the foundation of the de Rham cohomology theory

tity, can be rewritten as

$$\mathbf{dQ} = \partial_l Q_{[ijk]}\, dx^l \wedge dx^i \wedge dx^j \wedge dx^k = 0,$$

where $\partial_i \equiv \partial/\partial x^i$, while $\theta_{[ij\ldots]}$ denotes the skew-symmetric part of $\theta_{ij\ldots}$.

Similarly, the third equation (13) in components reads

$$\begin{aligned} \frac{1}{3!} Q_{ijk}\, dx^i \wedge dx^j \wedge dx^k &= \partial_k \alpha_{[ij]}\, dx^k \wedge dx^i \wedge dx^j, \qquad \text{or} \\ Q_{ijk} &= -6\partial_k \alpha_{[ij]}. \end{aligned}$$

The second equation (12) in components reads

$$\begin{aligned} \frac{1}{3!} \dot{Q}_{ijk}\, dx^i \wedge dx^j \wedge dx^k &= -\partial_k S_{[ij]}\, dx^k \wedge dx^i \wedge dx^j, \qquad \text{or} \\ \dot{Q}_{ijk} &= 6\partial_k S_{[ij]}. \end{aligned}$$

Finally, the first equation (11) in components reads

$$\begin{aligned} \frac{1}{2} \dot{\alpha}_{ij}\, dx^i \wedge dx^j &= (\partial_j J_i - \frac{1}{2} S_{ij})\, dx^i \wedge dx^j, \qquad \text{or} \\ \dot{\alpha}_{ij} &= 2\partial_j J_i - S_{ij}\,. \end{aligned}$$

In words, we have:

- The 2–form equation (11) defines the time derivative $\dot{\boldsymbol{\alpha}} = \frac{1}{2}\dot{\alpha}_{ij}\, dx^i \wedge dx^j$ of the dislocation density $\boldsymbol{\alpha}$ as the (negative) sum of the disclination current $\mathbf{S}$ and the curl of the dislocation current $\mathbf{J}$.

- The 3–form equation (12) states that the time derivative $\dot{\mathbf{Q}} = \frac{1}{3!}\dot{Q}_{ijk}\, dx^i \wedge dx^j \wedge dx^k$ of the disclination density $\mathbf{Q}$ is the (negative) divergence of the disclination current $\mathbf{S}$.

- The 3–form equation (13) defines the disclination density $\mathbf{Q}$ as the divergence of the dislocation density $\boldsymbol{\alpha}$, that is, $\mathbf{Q}$ is the *exact* 3–form.

- The Bianchi identity (14) follows from equation (13) by *Poincaré lemma* and states that the disclination density $\mathbf{Q}$ is conserved quantity, that is, $\mathbf{Q}$ is the *closed* 3–form. Also, every 4–form in 3D space is zero.

From these equations, we can derive two important conclusions:

1. Being the derivatives of the dislocations, brain's disclinations are higher–order tensors, and thus more complex quantities, which means that they present a higher risk for the severe TBI than dislocations — a fact which *is* supported by the literature (see review of existing TBI–models given in Introduction of [28]).

2. Brain's dislocations and disclinations are mutually coupled by the underlaying $SE(3)-$group, which means that we cannot separately analyze translational and rotational TBIs — a fact which *is not* supported by the literature.

For more medical details and references, see [28].

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